CHAOTIC OSCILLATION DURING VORTEX-INDUCED VIBRATION

Justin Leontini* & Mark Thompson**
*Centre for Mathematical and Information Science, Commonwealth Scientific and Industrial Research Organisation (CSIRO), Clayton, Victoria, 3168, Australia
**Fluids Laboratory for Aeronautical and Industrial Research (FLAIR), Department of Mechanical Engineering, Monash University, Clayton, Victoria, 3800, Australia

Summary Results are presented that show for a small range of flow speeds, the vortex-induced vibration response of an elastically-mounted cylinder is chaotic. This is in spite of the flow being restricted to being two-dimensional. Aside from observation of high amplitudes of response and disordered vortex configurations in the wake, the leading Lyapunov exponent is estimated from the numerical experiment data. The fact that this exponent is positive gives quantitative evidence for the fact that the response is chaotic.

INTRODUCTION AND BACKGROUND

Vortex-induced vibration of an elastically-mounted cylinder constrained to oscillate across the flow is governed by three parameters: the Reynolds number, $Re = UD/\nu$; the reduced velocity, $U^* = U/f_N D$; and the mass ratio, $m^* = m/m_f$. Here, U is the freestream velocity, D is the cylinder diameter, ν is the kinematic viscosity, f_N is the natural frequency of the cylinder, m is the mass of the cylinder, and m_f is the mass of fluid displaced by the cylinder.
Experiments such as those by Khalak & Williamson (1999) indicate there are two periodic regimes of response over a range of U^*. However, experiments by Hover *et al.* (1998) suggest that the higher-amplitude response regime is highly disordered.
Blackburn & Henderson (1996) and Leontini *et al.* (2006) conducted two-dimensional simulations that returned results qualitatively indicating that this higher-amplitude response regime is possibly chaotic.
The current results, where the largest Lyapunov exponent has been estimated from the data, quantitatively confirm that this response is genuinely chaotic.

METHODOLOGY

The simulations were conducted using a well-validated spectral-element method. Details of the implementation can be found in Thompson *et al.* (1996).
Lyapunov exponents measure the divergence of trajectories in phase space. The phase space is constructed of all the state variables of a system. Considering just the cylinder, there are three state variables; the transverse displacement y, the cylinder velocity, $\mathbf{V}_{cyl}$, and the lift force coefficient, C_L. It can be considered that the fluid effects can be "lumped" in the forcing term, resulting in a three-dimensional phase space. As the Lyapunov exponents measure divergence, they essentially measure the rate that points that start close to each other in phase space move away from each other. To first-order, this can be written as

$$d(t) = \exp \lambda t d_0 \qquad (1)$$

where d is the distance between two points close together in the phase space, λ is the Lyapunov exponent, and t should go to infinity. However, as data obtained from non-linear systems will eventually saturate (meaning higher-order terms, other than the first-order terms shown in the above equation will dominate), t must be limited. A good choice is something close to a primary period of the system. However, if it is chaotic, this can be difficult to choose *a priori*. For this study, the time period chosen was the timestep. The results obtained were then smoothed, obtaining essentially an average. Further details of the implementation can be found in Rosenstein *et al.* (1996).
Equation 1 can be re-arranged to read

$$\log(d(t)) - \log(d_0) = \lambda t \qquad (2)$$

Therefore, λ can be devised by plotting the differences between the logarithms of successive distances between two initially close points in the phase space, and measuring the gradient (or at least the gradient of an initially linear section, before higher-order terms have significant effect). A positive value of λ indicates that the response is chaotic.

RESULTS

The results presented here are from data obtained for values of the control parameters $Re = 200$, $m^* = 10$, and $U^* = 4.6$. These values result in almost the highest peak amplitude of y obtained when the flow is two-dimensional, exceeding $0.5D$ in one direction. The time history of the flow is far from periodic, as indicated from the images in figure 1. These images are a series of images of contours of vorticity at different times during the simulation. It is seen that the wake progresses from an ordered single-row configuration of vortices, to a double-row configuration of vortices, then into disorder. This disorder reduces the oscillation amplitude, and the process begins anew.

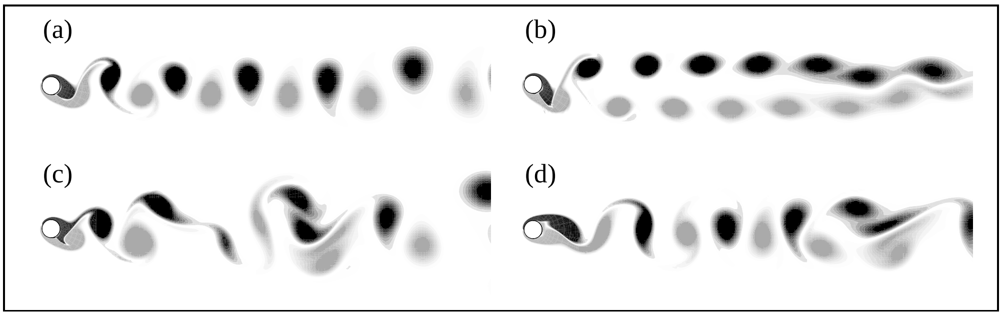

Figure 1. Snapshots of the wake for $U^* = 4.6$, when (a) $\tau = 300$, (b) $\tau = 350$, (c) $\tau = 416$, and (d) $\tau = 450$, demonstrating the evolution of the wake over time.

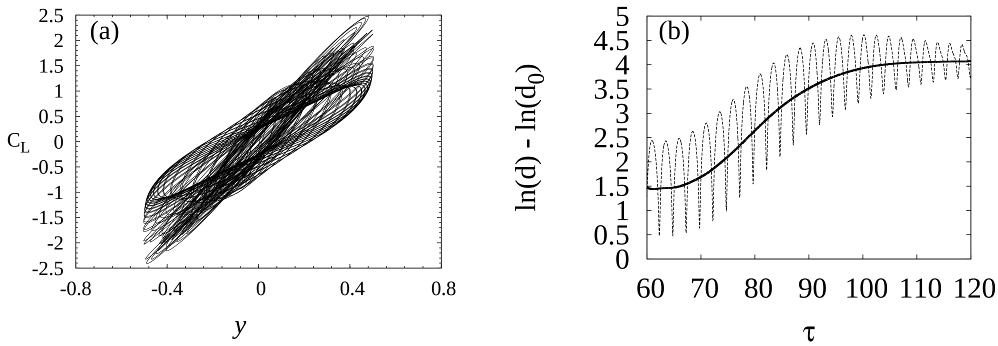

Figure 2. (a) A projection of the response in phase space, plotting C_L against y. This plot clearly demonstrates that the response is not quasiperiodic. (b) An example of the type of plot used to determine the leading Lyupanov exponent. The solid line is the running average of $\log(d(t)) - \log(d_0)$. The dotted line is the instantaneous value of the same quantity.

It could be thought that this growth and decay sequence results in simply a modulated oscillation, rendering the response quasi-periodic, but not chaotic. However, this is not the case, as is clear from inspection of figure 2a. This shows a projection of the phase space onto the lift-coefficient, displacement (C_L,y) plane. If the response was quasiperiodic, a spiral starting at the centre should be repeated over many times. However, it is shown that the trajectories do not repeat, passing close to each other but never repeating, another hallmark of chaotic response.

To quantify this chaotic response, an estimate of the Lyapunov exponent was calculated. This was done according the the process described in the methodology. This required plotting the difference of the logarithms of the distances apart of two points initially close over time, and identifying the gradient of the linear portion. An example of such a plot is given in figure 2b. This process was repeated a series of times, with different initial points. An average value of $\lambda = 0.14$ was obtained, where the measurements of λ were such that $0.069 < \lambda < 0.23$. While the spread of the data was relatively large, all indicated a positive value for λ, quantifying that the response is chaotic.

CONCLUSIONS

Two-dimensional simulations of an elastically-mounted cylinder in cross-flow have been performed. It was discovered that, for a particular value of U^*, the estimated Lyapunov exponent is positive, indicating that the response is truly chaotic, and not just the juxtaposition of two periodic solutions.

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