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STATE SELECTION IN TAYLOR VORTEX FLOW

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ABSTRACT

A numerical investigation was undertaken into state selection in Taylor vortex flow. The outer cylinder was assumed stationary. The inner cylinder Reynolds number was linearly increased in time from a fixed subcritical value to a fixed supercritical value over a finite ramp time and then held constant at the final Reynolds number. Different ramp times were considered ranging from impulsive increases to quasi-steady increases. For impulsive increases the preferred axial wavelength of the Taylor vortex flow was less than the critical wavelength. As the ramp time was increased the preferred axial wavelength increased toward the critical wavelength. For sufficiently large ramp times the preferred axial wavelength was always equal to the critical wavelength. A linear model was developed that modelled the change in preferred wavelength with ramp time. In the case of sufficiently large ramp times the model predicted that the amplitude of the mode with the critical wavelength grew to high amplitudes first. This mode then self-interacted and approached the form of steady Taylor vortex flow. Nonlinear effects were discussed with the addition of nonlinear terms in the amplitude equations.

INTRODUCTION

Taylor-Couette flow is the fluid bounded by two concentric rotating cylinders. Depending on the angular speeds of the two cylinders flows with different symmetries can be observed. We assume that the outer cylinder is stationary and the cylinders are infinite in their axial extent. We define an Reynolds number $Re = Vd/\nu$ where V is the inner cylinder speed, d is the gapsize of the cylinders and ν is the kinematic viscosity of the fluid. When Re is sufficiently small circular Couette flow is observed. When Re exceeds a critical value there is a transition to a steady Taylor vortex flow. Taylor vortex flow is axisymmetric and appears as a pairs of counter-rotating toroidal vortices periodically arranged in the axial direction.

Taylor vortex flow exhibits state nonuniqueness. In other words, Taylor vortex flows with different axial wavelengths can be achieved at the same final Reynolds number. Which axial wavelength is selected depends on the way in which the final state is approached.

Experiments by Burkhalter & Koschmieder (1974) for impulsive increases of the inner cylinder Reynolds number showed that the preferred axial wavelengths

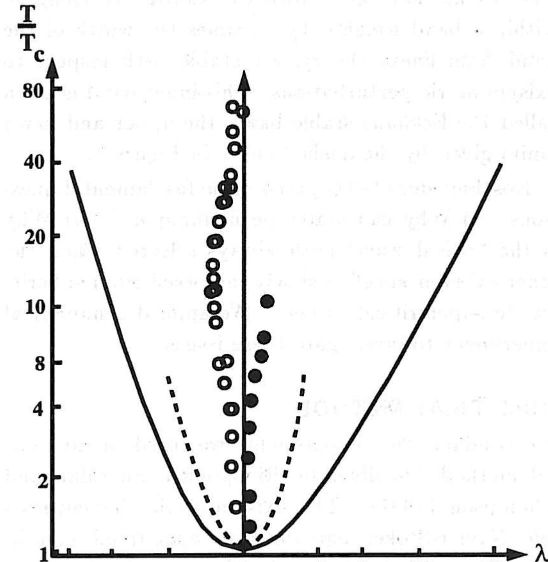


Figure 1: Stability diagram for Taylor vortex flow for $\eta = 0.727$, $\mu = 0$. Taylor number versus axial wavelength. The solid line is the neutral curve from the linear stability of steady circular Couette flow. The dashed line shows the stability boundary from a weakly nonlinear analysis by Kogelman & DiPrima (1970). The open circles represent Taylor vortex flows states observed for sudden start experiments by Burkhalter & Koschmieder (1974). The solid circles represent Taylor vortex flow states observed for an experiment where the annulus was filled with fluid after inner cylinder was rotating at fixed speed. From Burkhalter & Koschmieder (1974).

were smaller than critical wavelength, illustrated by the open circles in Figure 1. Also, experiments showed that very slow increases in the inner cylinder speed from subcritical to supercritical Reynolds numbers always resulted in a Taylor vortex flow with the critical wavelength. In Figure 1, the critical wavelength is $\lambda_c = 2.0$. Koschmieder (1993) found that different inner cylinder acceleration rates from fixed subcritical to fixed supercritical Reynolds numbers resulted in different Taylor vortex flows being preferred with wavelengths between those obtained by impulsive increases and the critical wavelength.

In Figure 1, the Taylor number is related to Reynolds number from the equation $T/T_c = (Re/Re_c)^2$. The outer curve is the neutral curve from a linear stability analysis of steady circular Couette flow. For a particular Reynolds number there is a band of axial wavelengths, with upper and lower limits given by the outer curve, that will grow exponentially. Perturbations with axial wavelengths outside this band will decay exponentially.

Kogelman & DiPrima (1970) undertook a weakly nonlinear stability analysis of Taylor vortex flow. They found that Taylor vortex flows with wavelengths within a band roughly $1/\sqrt{3}$ times the width of the band from linear theory, are stable with respect to axisymmetric perturbations. This inner band is often called the Eckhaus stable band, the upper and lower limits given by the dashed curve in Figure 1.

Koschmieder (1993) posed some fundamental questions: (i) Why can states be nonunique ? (ii) Why is the critical wavelength always selected when the inner cylinder speed is slowly increased from subcritical to supercritical values ? We applied a numerical experiment to investigate these issues.

NUMERICAL METHOD

To conduct the experiment we used a numerical method described by Rigopoulos, Sheridan and Thompson (1997). The axisymmetric, incompressible Navier-Stokes equations in cylindrical coordinates were solved numerically using a spectral method and with the use of operator splitting. The method was tested against known values of growth rate for Taylor vortex flow and was shown to give second-order time-accuracy. The height of the cylinders was assumed infinite and so periodic axial boundary conditions were applied. The velocity and pressure were represented with a Fourier approximation in the axial direction and a Chebyshev approximation in the radial direction.

NUMERICAL EXPERIMENT

We considered the case with radius ratio $\eta = 0.727$ in order to relate results to those by Burkhalter & Koschmieder (1974). A stationary outer cylinder was assumed. The inner cylinder Reynolds number was linearly increased in time from an initial subcritical value, $Re_i = 70$, to a final supercritical value, $Re_f = 116.67$, over a finite ramp time, T, and then held fixed at Re_f. A large aspect ratio of $\Gamma = 20.0286$ (ten times the critical wavelength value) was used in order to allow the nonlinear interaction of many axisymmetric modes and thus simulate the state selection process. A 324×33 Fourier-Chebyshev grid and a timespacing of $\Delta t = 0.1$ were used. A number of different simulations were conducted for different ramp times T. In each simulation, initial conditions of circular Couette flow plus a random perturbation of the order 10^{-4} was applied. A random perturbation was considered since this is typical in a physical experiment. The random number generation sequence was kept the same for each simulation to ensure that initial conditions were fixed.

RESULTS

In Figure 2 is shown results for amplitude of the modes, $A_\lambda(t)$, versus time for four different ramp times. The modes are for five discrete axial wavelengths in the vicinity of the critical wavelength: 1.54066, 1.66905, 1.82078, 2.00286 and 2.2254 with corresponding Fourier indices 13, 12, 11, 10 and 9, respectively. The critical wavelength is $\lambda_c = 2.00286$. with critical Reynolds number $Re_c = 82.8$ for the slow ramp case of $T = 99.61$. We calculated $A_\lambda(t)$ from the magnitude of the Fourier transform of the radial component of velocity at the radial centre of the gap.

Figure 2 shows that the λ_c is preferred for sufficiently long ramp times, as illustrated for $T = 99.61$ and $T = 149.42$. For sudden starts, $T = 0$, the preferred wavelength $\lambda = 1.66905$ is less than λ_c. The selected wavelength changes from 1.66905 to 1.82078 to 2.00286 for progressively longer ramp times. The wavelengths of these three Taylor vortex flow states lie within the Eckhaus stable band. These features agree with results by Koschmieder (1993).

LINEAR MODEL

While the amplitudes of the modes are small one has to solve the amplitude equation

$$\frac{dA_\lambda(t)}{dt} = \sigma_\lambda(t) A_\lambda(t) \quad (1)$$

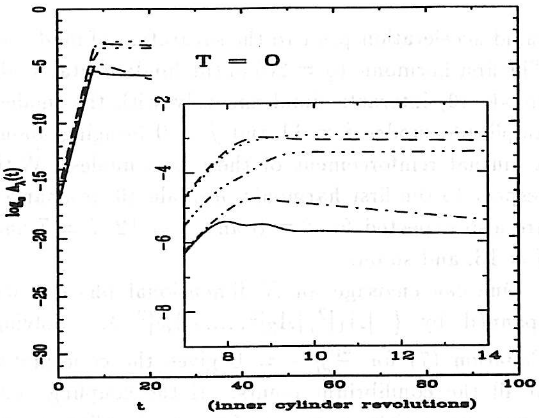
where $\sigma_\lambda(t)$ is the instantaneous exponential growth rate which is a linear function of the instantaneous Reynolds number $Re(t)$. It is given by

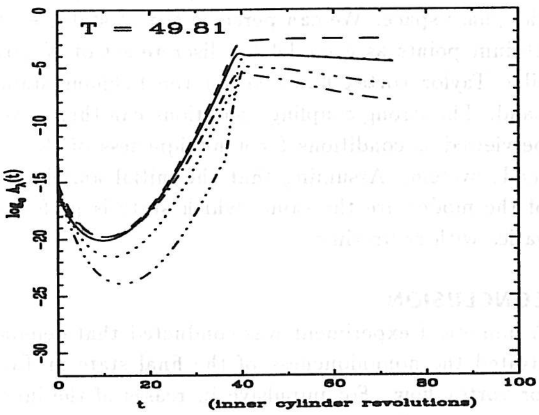
$$\sigma_\lambda(t) = \begin{cases} -\sigma_{o,\lambda} + K_\lambda \frac{Re_f - Re_i}{T} t & \text{if } t \leq T \\ \sigma_{f,\lambda} & \text{if } t > T \end{cases} \quad (2)$$

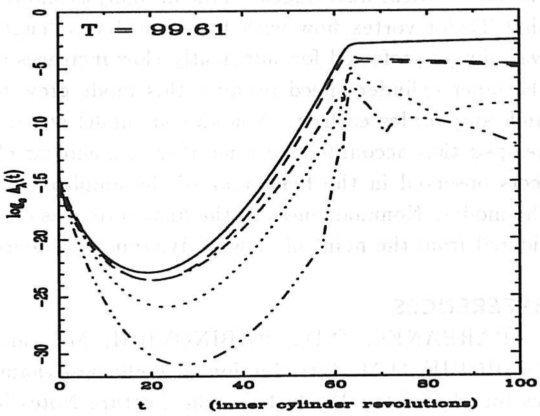
where

$$\sigma_{o,\lambda} = -K_\lambda (Re_i - Re_{c,\lambda}) \quad (3)$$

and







$$\sigma_{f,\lambda} = K_\lambda(Re_f - Re_{c,\lambda}) \qquad (4)$$

The factor K_λ is the cofactor of proportionality and is mode dependent. The solution to Equation (1) is then

$$A_\lambda(t) = \begin{cases} A_\lambda(0)e^{-\sigma_{o,\lambda}t+\frac{1}{2}K_\lambda\frac{Re_f-Re_i}{T}t^2} & \text{for } t \leq T \\ A_\lambda(0)e^{\frac{1}{2}(\sigma_{f,\lambda}-\sigma_{o,\lambda})T+\sigma_{f,\lambda}(t-T)} & \text{for } t > T \end{cases} \qquad (5)$$

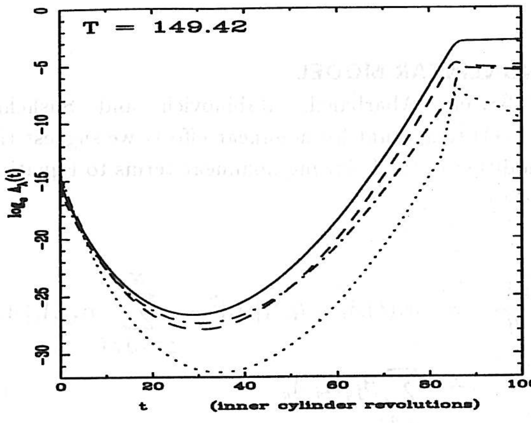


Figure 2: $A_\lambda(t)$ for four different ramp times T.

— · — ·	$\lambda = 2.2254$	(mode 9)
———	$\lambda_c = 2.00286$	(mode 10)
– – – –	$\lambda = 1.82078$	(mode 11)
·······	$\lambda = 1.66905$	(mode 12)
– · · · –	$\lambda = 1.54066$	(mode 13)

The log of the amplitude of the modes will therefore vary quadratically in time during the ramp and Equation (5) models the behaviour of the modes in Figure 2 while the amplitudes are small.

Now, consider the time, t^*_λ it takes for the amplitude of each mode λ to grow to a particular higher amplitude A_H, where nonlinear effects begin to become important. In Figure 2 for $T = 49.81$, 99.61 and 149.42, A_H would be approximately where the amplitude of the preferred mode ceases to vary quadratically with time. Figure 2 for $T = 49.81$, 99.61 and 149.42 implies that $T > t^*_\lambda$. Equation (5) for $t \leq T$ leads to

$$t^*_\lambda \approx 2\left(\frac{Re_{c,\lambda} - Re_i}{Re_f - Re_i}\right)T \qquad (6)$$

for T sufficiently large. Equation (6) predicts that for $T = 149.42$ the mode with the critical wavelength has $t^*_{\lambda_c} \approx 82$, which is consistent with Figure 2 for $T = 149.42$.

The linear model thus predicts that for sufficiently long ramp times the mode with the least time to reach high amplitudes is the mode which has the minimum critical Reynolds number (i.e. the mode with the critical wavelength). This mode then self-interacts and approaches the form of steady Taylor vortex flow.

NONLINEAR MODEL

Following Abarbanel, Rabinovich and Sushchik (1993) to account for nonlinear effects we suggest the addition of the following nonlinear terms to Equation (1):

$$\frac{dA_i}{dt} = \sigma_i(t)A_i - l_i A_i|A_i|^2 - \sum_{j=1,j\neq i}^{N} \alpha_{ij} A_i |A_j|^2 + \sum_{j,q} \beta_{jq} A_j^* A_q^* \qquad (7)$$

for N modes where the asterisk denotes complex conjugate. The subscripts now denote Fourier indices.

In Equation (7) the first cubic term accounts for the self-interaction of mode i. The second cubic term accounts for the coupling of mode i with other modes j. The quadratic term models resonant three-wave interactions. The index q is taken over all harmonics and is such that the resonance condition $i + j = q$ is satisfied.

Consider the Eckhaus mechanism of instability of a Taylor vortex flow with a fundamental mode k and first harmonic $q = 2k$. When there are side-band perturbations with modes i and j such that $i + j = 2k$, these perturbations resonate with the first harmonic and mutually reinforce each other, destabilizing the Taylor vortex flow. However, when we consider a Taylor vortex flow within the Eckhaus stable band, the resonances still occur but they are not strong enough to destabilize the flow.

In Figure 2 for $T = 49.81$, $T = 99.61$ and $T = 149.42$ there are regions of rapid acceleration for the lower amplitude modes prior to the saturation of the preferred mode. These regions of rapid acceleration are due to resonant three-wave interactions.

In Figure 2 for $T = 49.81$ the preferred mode is mode 11. During the ramp the log of the amplitude of mode 12 varies quadratically with time whilst its amplitude is small, followed by a rapid acceleration prior to the saturation of mode 11. The first harmonic ($q = 22$) of the fundamental mode (mode 11) interacts simultaneously with the smaller amplitude modes $i = 10$ and $j = 12$, bringing about a mutual reinforcement or resonance of these two modes (since the condition under which the resonance occurs is satisfied, namely $i + j = q$). One also expects mutual reinforcements of modes $i = 9$ and $j = 13$, $i = 8$ and $j = 14$, and so on. These resonances are a result of interactions with the first harmonic of the mode 11. These are also manifested as rapid accelerations prior to the saturation of the fundamental mode.

Also, in Figure 2 for $T = 99.61$ and $T = 149.42$ the preferred mode is mode 10. During the ramp the log of the amplitude of mode 9 varies quadratically with time whilst its amplitude is small, followed by a rapid acceleration prior to the saturation of mode 10. The first harmonic ($q = 20$) of the fundamental mode (mode 10) interacts simultaneously with the smaller amplitude modes $i = 11$ and $j = 9$ bringing about a mutual reinforcement of these two modes. With respect to the first harmonic of mode 10, resonances are also expected for $i = 8$ and $j = 12$, $i = 7$ and $j = 13$, and so on.

One can envisage an N-dimensional phase space spanned by $\{ |A_1|^2, |A_2|^2, ..., |A_N|^2 \}$. Solving Equation (7) for $\frac{d|A_i|^2}{dt} = 0$ gives the coordinates of all the equilibrium points. If the coupling constants α_{ij} satisfy some set of strong coupling conditions then we obtain N stable equilibrium points in the phase space. We can perceive the N stable equilibrium points as a model for discrete set of N possible Taylor vortex flows within the Eckhaus stable band. The strong coupling conditions can themselves be viewed as conditions for nonuniqueness of the N-mode system. Assuming that the initial amplitudes of the modes are the same, which state is preferred varies with ramp time.

CONCLUSION

A numerical experiment was conducted that demonstrated the nonuniqueness of the final state in Taylor vortex flow. For impulsive increases of the inner cylinder speed the preferred axial wavelength was less than the critical wavelength. A linear analysis showed that Taylor vortex flow with the critical wavelength was always preferred for sufficiently slow increases of the inner cylinder speed because this mode grew to high amplitudes earliest. A nonlinear model was developed that accounted for a number of nonlinear effects observed in the behaviour of the amplitude of the modes. Nonuniqueness of the final state was considered from the point of view of dynamical systems.

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