On quasiperiodic and subharmonic Floquet wake instabilities

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The physical characteristics of bifurcated states in systems with inherent symmetry are constrained in ways that those in systems with broken symmetry are not. Here we examine the issue of quasiperiodic versus subharmonic instability modes of time-periodic laminar wakes, and how the relationship between them is influenced by weak symmetry breaking. The examples used are the vortex street wake of a circular cylinder, where symmetry is broken by distorting the cylinder into a ring, and the wake of a square cylinder, where symmetry is broken by a small fixed rotation of the cylinder about its axis. In both cases the symmetric wakes exhibit a quasiperiodic instability mode, with a pair of complex-conjugate Floquet multipliers and which manifests as a traveling wave. As symmetry is broken these multipliers migrate continuously to the real axis, coalesce, and split into a pair of subharmonic multipliers that move apart along the negative real axis. This behavior resolves an apparent dichotomy between the previously established theoretical results and numerical predictions for the symmetric wake systems, and the predictions and experimental observations for systems with weakly broken symmetry. © *2010 American Institute of Physics*.
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The study of three dimensionality of bluff body wakes has a long history, with concentrated theoretical attention following detailed experimental examinations of the circular cylinder wake at Reynolds numbers near the onset of secondary instability, as summarized by Williamson.[1] Numerical methods were brought to bear in predicting the primary and secondary (Floquet) instabilities of the circular cylinder wake, as exemplified by the works of Jackson[2] and Barkley and Henderson.[3] The Floquet analysis generally agreed well with the experimental observations of two distinct secondary instability modes (the long wavelength mode A and the shorter wavelength mode B) in showing that these bifurcate successively from the two-dimensional base states as Reynolds number $\mathrm{Re}=U_\infty d/\nu$ is increased; the predicted critical Reynolds numbers, spanwise wavelengths, and mode shapes were close to the experimental observations.

In related theoretical developments, attention was paid to the underlying symmetries of the two-dimensional base states in these problems, and the implications this held for the ways in which symmetry could be broken in secondary instabilities of the vortex street wakes. The two-dimensional time-periodic wake of a circular cylinder, or any two-dimensional geometry with a reflection symmetry with respect to the incident flow direction, such as a square, a diamond, a symmetrical airfoil, or a flat plate normal to or aligned with the flow, has the property that its shape remains the same under temporal evolution by a half-period, combined with a reflection about a body axis aligned with the flow direction. This symmetry is illustrated for the wake of a square cylinder in Figs. 1(a) and 1(b). Such a spatiotemporal symmetry is a member of the Z_2 symmetry group, i.e., a generalized reflection.

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Evolution of the wake by a full period T in time corresponds to two of these operations, and this squaring has important implications for possible ways in which symmetries can be broken through bifurcations, as first pointed out by Swift and Wiesenfeld:[4] period-doubling bifurcations are suppressed as generic events. This suppression may be readily understood by considering the loci of Floquet multipliers in the complex plane, where a bifurcation corresponds to either a single real multiplier or a complex-conjugate pair emerging from the unit circle. The generic ways this may occur are (i) along the positive real axis as a synchronous bifurcation; (ii) along the negative real axis as a period-doubling or subharmonic bifurcation; (iii) a complex-conjugate pair evolving along a generically curved pair of paths that have reflection symmetry about the real axis as a Neimark–Sacker bifurcation, i.e., the generalization of a Hopf bifurcation to a limit cycle base state.[5] The operation of squaring removes the subharmonic bifurcation as a generic possibility on the full period, since period doubling on the $T/2$ mapping is synchronous when taken on period T.

When considering the secondary bifurcations of wakes the translation/reflection symmetry of the wake along the span must be considered too. For bifurcated states that have a spanwise waviness, this is an $O(2)$ symmetry group, meaning that the complete symmetry group of the two-dimensional wakes (and related systems[6]) is $Z_2\times O(2)$. The theoretical development of the possible symmetry breaking bifurcations for this symmetry group has been previously pursued[7] and applied to the study of the circular cylinder wake and the square cylinder wake.[8] Generically there are two types of T-synchronous secondary instabilities in these systems; ones that preserve and others that break the underlying symmetry group (in the cylinder wake these correspond, respectively, to modes A and B), as well as two types of quasiperiodic instabilities, one of which manifests as

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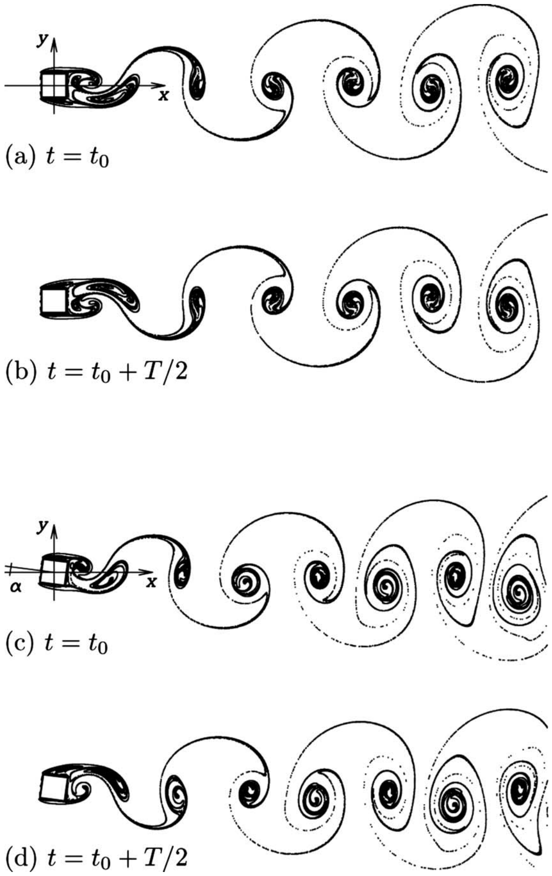

FIG. 1. Computed locations of marker particles for a two-dimensional square cylinder wake at $\mathrm{Re}=U_\infty d/\nu=225$. [(a) and (b)] With the square aligned to the coordinate axes the wake has a spatiotemporal symmetry such that it is identical after a temporal evolution by $T/2$ and a reflection about the x axis. [(c) and (d)] With the square rotated about its axis through angle α (here $\alpha=7.5°$) this spatiotemporal symmetry is lost. When $t=t_0$ [i.e., for (a) and (c)], lift force is maximum.

modulated traveling waves and the other which manifests as standing waves. Under weakly nonlinear evolution, only one of the possible pair of quasiperiodic modes will be observed,[9] and in the systems thus far considered this has been the traveling wave state.

Quasiperiodic three-dimensional instabilities of the circular cylinder wake were first predicted by Barkley and Henderson.[3] Subsequent study[8] showed these to be the third state to bifurcate from two-dimensional base states with increasing Reynolds number, with $\mathrm{Re}_c=377$, and to manifest as traveling waves in the weakly nonlinear case. Much the same behavior was demonstrated for the wake of the square cylinder,[8,10] while for a related problem with the same symmetry group but two control parameters,[6] again two synchronous modes and a traveling wave mode were predicted as three-dimensional bifurcations from two-dimensional T-periodic states. All these findings were in full accord with the theoretical development.[7]

In the meantime, studies of cases which were apparently related but which broke perfect Z_2 symmetry had been initiated, originally for ring wakes as a model for the circular cylinder wake but without experimentally problematic end effects.[11] These led into numerical Floquet analysis[12,13] of ring wakes of moderate to low aspect ratio $A=D/d$, where D is the mean ring diameter and d the diameter of its cross section. The three-dimensional instabilities of ring wakes again were found to provide two synchronous modes of differing wavelength, related to the cylinder wake A and B modes. However, instead of a quasiperiodic mode, these analyses also predicted a third mode ("mode C") of subharmonic type,[14] and this was verified to occur experimentally.[15]

More recently, Floquet analysis of the wake of a square cylinder was revisited, focusing on the effect of angle of incidence variation on secondary instabilities.[16] When the angle of incidence $\alpha=0$ (i.e., the leading face of the square is normal to the incident flow direction), this study found two synchronous modes and a quasiperiodic mode, the third to bifurcate from two-dimensional base states with increasing Re, in agreement with previous work.[8,10] At $\alpha=7.5°$, the first nonzero value studied, this quasiperiodic mode is replaced by a subharmonic one. At moderate angles of incidence, $\alpha=12°-26°$, the subharmonic mode is the first to bifurcate from the two-dimensional base state as Reynolds number is increased. Figures 1(c) and 1(d) illustrate the breakage of perfect spatiotemporal symmetry for the two-dimensional wake of a square cylinder at Re=225, $\alpha=7.5°$.

The two-dimensional wakes of the ring and the rotated square section are examples where the Z_2 spatiotemporal symmetry of the wakes of the circular and square cylinders is broken, albeit via different physical mechanisms, apparently with the same outcome: the replacement of a quasiperiodic instability mode with a subharmonic mode. This result serves as the motivation for the present work, which is to examine how transition occurs as symmetry is broken. We use the two examples already introduced, the ring and the rotated square. These are good examples in part precisely because the physical mechanism is distinct in the two cases: introduction of curvature in the first and a simple rotation that leaves a Cartesian geometry in the second. However, in theoretical terms, they have commonality, in that these distortions each break the Z_2 spatiotemporal symmetry of the original problem, and introduce a single control parameter as a measure of broken symmetry, aspect ratio A in the case of the ring and angle of incidence α in the case of the rotated square.

The codes used to carry out Floquet analysis for these problems are two independent implementations of Krylov-type timestepper-based eigensystem methods,[17,18] where the underlying discretization is nodal Galerkin spectral elements on mapped quadrilateral elements, combined with time-split integration based on backward differencing.[19] The cylindrical-coordinate technique used to compute ring wakes has been described in detail elsewhere;[20] the Cartesian-coordinate version used to compute cylinder wakes is a straightforward simplification. The same codes were used in the two studies[8,16] that formed the starting points for the present work, as well as many others. We have checked that the codes return identical results, and we have retained the mesh geometries employed in the two earlier studies.

Starting with the circular cylinder wake, the quasiperiodic mode is the third to bifurcate from the base state[8] with $\mathrm{Re}_c=377$ and a spanwise wavenumber $\beta_c=2\pi d/\lambda=3.5$. In the present study, the Reynolds number was held fixed at

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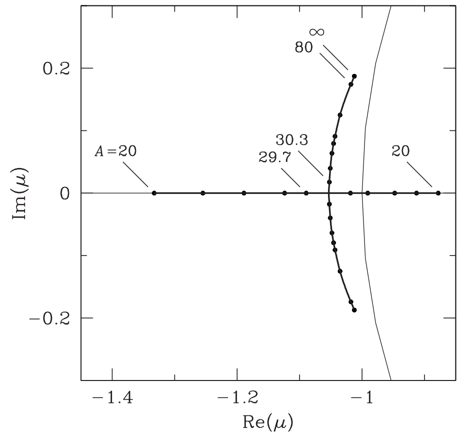

FIG. 2. Floquet multiplier locus with variation in ring aspect ratio $A=D/d$ shown with a portion of the unit circle in the complex plane for the circular cylinder/ring wake at Re=380, illustrating the evolution of a quasiperiodic mode into a pair of subharmonic modes.

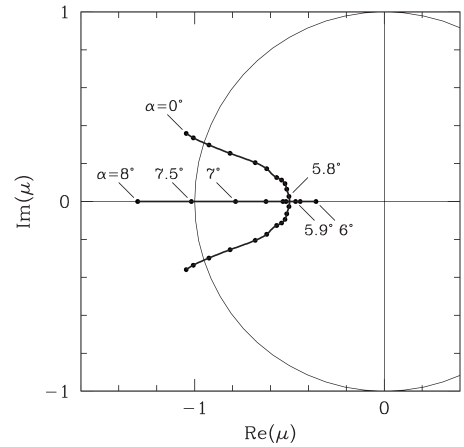

FIG. 3. Floquet multiplier locus with variation in angle of attack α for the rotated square cylinder wake at Re=225, $\lambda/d=2.4166$, compared with a portion of the unit circle in the complex plane.

Re=380. The computations were carried out on a mesh with cross-flow dimensions $\pm 10d$, meaning that the minimum aspect ratio attempted is $A=20$ (when one edge of the domain touches the axis of the cylindrical coordinate system) with no required upper limit ($A=\infty$ corresponding to the Cartesian infinite cylinder). Also fixed in the present study was the spanwise/circumferential wavelength $\lambda/d=2\pi/\beta_c=1.7952$, and the aspect ratio was adjusted such that an integer number of waves with this length fit around the circumference of the ring, giving $A=2\beta/\beta_c$, where β here is an integer and $\beta_c=3.5$, the Cartesian-coordinate value.

Figure 2 shows the detail of resulting Floquet multiplier locus. At $A=\infty$ (a straight circular cylinder) the quasiperiodic multipliers lie outside the unit circle as a complex-conjugate pair, i.e., $\mu=(|\mu|, \angle \pm \theta)$ with $|\mu|>1$, and where the secondary period of the quasiperiodic state is given by $T_s/T=2\pi/\theta>2$. As A is decreased and Z_2 symmetry of the base state is broken, the multipliers migrate toward the negative real axis with very slightly increasing moduli, and the associated secondary periods T_s of the quasiperiodic states decrease. The complex-conjugate pair meets and coalesces on the negative real axis when $A=30$. At this stage the secondary period $T_s=2T$, i.e., the bifurcated solutions are period doubled or subharmonic. With further decrease in A, the two multipliers split and migrate in opposite directions along the real axis, one receding inside the unit circle and the other moving increasingly rapidly away from the origin. Thus, the initial effect of symmetry breaking is to decrease the secondary period but without suppressing quasiperiodicity, while the ultimate effect is to destabilize one of the associated modes once period doubling has been achieved, following a finite degradation of symmetry.

The secondary instability behavior of the square cylinder wake in the symmetric $\alpha=0$ case is broadly similar to that of the circular cylinder: the quasiperiodic mode is the third to bifurcate from the basic state and this occurs at $\mathrm{Re}_c=215$ with a spanwise wavelength $\lambda/d=2.65$.[16] As for the study of the ring wake, we focus on the behavior of this mode. The Reynolds number and spanwise wavelength are held fixed (at Re=225, $\lambda/d=2.4166$), and the symmetry control parameter, here α, is varied. We note that as the square is rotated, the Reynolds number based on the projected cross-flow dimension would increase by a factor up to $2^{1/2}$ at $\alpha=\pi/4$; however, this variation is not accounted for $\mathrm{Re}=U_\infty d/\nu=\mathrm{const}$, where d is the length of a side.

Figure 3 shows the multiplier locus associated with the quasiperiodic/subharmonic mode for the rotated square cylinder. At $\alpha=0$, the quasiperiodic mode is slightly unstable. As symmetry is broken for $\alpha>0$, the associated mode becomes more stable initially, as the complex-conjugate multiplier pair recedes inside the unit circle and migrates toward the negative real axis, and the associated secondary period of the quasiperiodic mode reduces toward $2T$. At $\alpha=5.85°$, the multipliers coalesce on the negative real axis, while for larger α the pair splits and each moves in opposite directions along the negative real axis. Eventually at $\alpha\approx 7.5°$, one of this pair of subharmonic modes bifurcates from the two-dimensional state when $|\mu|>1$.

We note that in the analysis above we have kept the spanwise wavelengths of the three-dimensional instabilities constant. The most-amplified wavelength for quasiperiodic/subharmonic instability will vary slightly as symmetry is broken, but this effect is comparatively weak for the small variations considered, and the wavelengths employed fall near the center of the range for these modes across the parameter regimes considered.[8,12,16] Thus we are confident that our findings may principally be attributed to Z_2-symmetry breaking of the base flows.

This study has concentrated on the issue of how quasiperiodic secondary instabilities in two wake flows with spatiotemporal symmetry group $Z_2\times O(2)$ are affected by symmetry breaking. The behavior is very similar in the two cases: breaking symmetry produces a migration of a complex-conjugate pair of multipliers toward the negative real axis, where at some finite value of symmetry-breaking control parameter they coalesce on the real axis and split to produce a pair of subharmonic modes. One of these new modes is destabilized by further symmetry breaking, while the other moves toward the origin and becomes more stable.

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The observed behavior is in agreement with the theoretical results for bifurcations in symmetric systems:[4,7] Z_2 spatiotemporal symmetry of the two-dimensional wakes of the circular and the unrotated square cylinder acts to suppress subharmonic Floquet modes, leaving quasiperiodic modes as an alternative to synchronous modes. We have shown that as symmetry is broken, there is no discontinuous change to subharmonic states, but rather a finite amount of distortion brings the multipliers for these modes to the negative real axis where they coalesce to form a pair of subharmonic modes. Further distortion away from the symmetric base state acts to destabilize one of these two subharmonic modes. However, we note that another route for production of subharmonic modes could be genesis at the origin as soon as symmetry is broken, followed by a locus straight along the negative real axis, leaving quasiperiodic modes in existence even as a subharmonic mode bifurcates from the two-dimensional basic state.

Our findings resolve an apparent dichotomy between quasiperiodic versus subharmonic wake instability modes: subharmonic modes do not generically arise if the two-dimensional state has Z_2 spatiotemporal symmetry, as illustrated in Figs. 1(a) and 1(b). We have demonstrated one possible route to subharmonic instability as symmetry is broken [as shown in Figs. 1(c) and 1(d)] and also that in this scenario, breakage of symmetry is ultimately destabilizing.

[1] C. H. K. Williamson, "Vortex dynamics in the cylinder wake," Annu. Rev. Fluid Mech. **28**, 477 (1996).
[2] C. P. Jackson, "A finite-element study of the onset of vortex shedding in flow past variously shaped bodies," J. Fluid Mech. **182**, 23 (1987).
[3] D. Barkley and R. D. Henderson, "Three-dimensional Floquet stability analysis of the wake of a circular cylinder," J. Fluid Mech. **322**, 215 (1996).
[4] J. W. Swift and K. Wiesenfeld, "Suppression of period doubling in symmetric systems," Phys. Rev. Lett. **52**, 705 (1984).
[5] Y. A. Kuznetsov, *Elements of Applied Bifurcation Theory*, 2nd ed. (Springer-Verlag, Berlin, 1998).
[6] H. M. Blackburn and J. M. Lopez, "The onset of three-dimensional standing and modulated travelling waves in a periodically driven cavity flow," J. Fluid Mech. **497**, 289 (2003).
[7] F. Marques, J. M. Lopez, and H. M. Blackburn, "Bifurcations in systems with Z_2 spatio-temporal and $O(2)$ spatial symmetry," Physica D **189**, 247 (2004).
[8] H. M. Blackburn, F. Marques, and J. M. Lopez, "Symmetry breaking of two-dimensional time-periodic wakes," J. Fluid Mech. **522**, 395 (2005).
[9] J. D. Crawford and E. Knobloch, "Symmetry and symmetry-breaking bifurcations in fluid dynamics," Annu. Rev. Fluid Mech. **23**, 341 (1991).
[10] H. M. Blackburn and J. M. Lopez, "On three-dimensional quasi-periodic Floquet instabilities of two-dimensional bluff body wakes," Phys. Fluids **15**, L57 (2003).
[11] T. Leweke and M. Provansal, "The flow behind rings: Bluff body wakes without end effects," J. Fluid Mech. **288**, 265 (1995).
[12] G. J. Sheard, M. C. Thompson, and K. Hourigan, "From spheres to circular cylinders: The stability and flow structures of bluff ring wakes," J. Fluid Mech. **492**, 147 (2003).
[13] G. J. Sheard, M. C. Thompson, and K. Hourigan, "From spheres to circular cylinders: Non-axisymmetric transitions in the flow past rings," J. Fluid Mech. **506**, 45 (2004).
[14] G. J. Sheard, M. C. Thompson, and K. Hourigan, "Subharmonic mechanism of the mode C instability," Phys. Fluids **17**, 111702 (2005).
[15] G. J. Sheard, M. C. Thompson, K. Hourigan, and T. Leweke, "The evolution of a subharmonic mode in a vortex street," J. Fluid Mech. **534**, 23 (2005).
[16] G. J. Sheard, M. J. Fitzgerald, and K. Ryan, "Cylinders with square cross-section: Wake instabilities with incidence angle variation," J. Fluid Mech. **630**, 43 (2009).
[17] L. S. Tuckerman and D. Barkley, in *Numerical Methods for Bifurcation Problems and Large-Scale Dynamical Systems*, edited by E. Doedel and L. S. Tuckerman (Springer-Verlag, Berlin, 2000), pp. 453–566.
[18] D. Barkley, H. M. Blackburn, and S. J. Sherwin, "Direct optimal growth analysis for timesteppers," Int. J. Numer. Methods Fluids **57**, 1435 (2008).
[19] G. E. Karniadakis, M. Israeli, and S. A. Orszag, "High-order splitting methods for the incompressible Navier–Stokes equations," J. Comput. Phys. **97**, 414 (1991).
[20] H. M. Blackburn and S. J. Sherwin, "Formulation of a Galerkin spectral element-Fourier method for three-dimensional incompressible flows in cylindrical geometries," J. Comput. Phys. **197**, 759 (2004).

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