Microscopic bifurcation analysis for long-wave in-plane buckling of elastic square honeycombs

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Summary

In this study, microscopic buckling of elastic square honeycombs subject to in-plane compression is analyzed using a two-scale theory of the up-dated Lagrangian type. The theory allows us to analyze microscopic bifurcation and post-bifurcation behavior of periodic cellular solids. Cell aggregates are taken to be periodic units so that we can discuss the dependence of buckling stress on periodic length. Then, it is shown that microscopic buckling occurs at a lower compressive load as periodic length increases, and that long-wave buckling occurs just after the onset of macroscopic instability if the periodic length is sufficiently long. It is further shown that the macroscopic instability is of the shear type, leading to a simple formula to evaluate the lowest in-plane buckling stress of elastic square honeycombs.

Introduction

When cellular solids are subject to compression, buckling may occur in cell walls and edges. This instability is called microscopic buckling. Such microscopic instability may cause macroscopic instability. In analyzing the microscopic instability of periodic cellular solids, one must note the following feature: The eigenmode of microscopic bifurcation in periodic cellular solids is represented by the Bloch wave and thus can have a longer periodic length than unit cells; consequently, if the periodic length is infinite, the onset of microscopic bifurcation can be identified with the start of macroscopic instability [1]. The above mentioned feature is important and has brought about interest in analyzing the periodic length-dependent, or long-wave, microscopic bifurcation in cellular solids [2-5].

Square honeycombs made of ceramics are now used worldwide for catalytic converters. Metallic square honeycombs are also being developed as one of the ordered metallic cellular solids [6]. Square honeycombs are thus one of the important cellular solids, and their in-plane elastic properties and buckling stresses are now available in the literature [7]. Long-wave bifurcation, however, has not been evaluated for the in-plane buckling of square honeycombs. They are, therefore, worth examining for possible long-wave in-plane buckling.

In this study, long-wave bifurcation will be demonstrated for elastic square honeycombs subject to in-plane uniaxial compression. To this end, their in-plane buckling and post-buckling behavior will be analyzed using the two-scale theory and the bifurcation condition developed by the present authors [5,8]. Several cell-aggregates consisting of different numbers of cells will be assumed as periodic units. The results of analysis will be discussed to show that long-wave bifurcation occurs after the onset of

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macroscopic instability. Then, by utilizing the macroscopic instability condition, a simple formula will be derived to evaluate the lowest in-plane buckling stress of elastic square honeycombs.

Theory

First, we briefly describe the updated Lagrangian type two-scale theory developed by Ohno et al. [8] and Okumura et al. [5].

If an infinite, periodic body B with a unit cell *Y* is subject to macroscopically uniform stress or strain, the microscopic deformation of B can initially satisfy the so-called *Y*-periodicity, in which *Y* is the unit of periodicity. However, if microscopic bifurcation occurs, the *Y*-periodicity may break down, resulting in a larger periodic unit than *Y*. To represent such an enlarged periodic unit, we consider a cell aggregate consisting of *k* unit cells, *kY*, and we take Cartesian coordinates x_i $(i = 1, 2, 3)$ for *kY* in the current configuration of B . Hereafter, $(\ )_{,i}$ will indicate the differentiation with respect to x_i, and $(\dot{\ })$ will designate the material derivative with respect to time *t*. Moreover, the summation convention will be used, unless otherwise stated.

If the *kY*-periodicity of microscopic deformation prevails in B , velocity $\dot{u}_i$ can be divided into macroscopic part $\dot{u}_i^0$ and microscopic, *kY*-periodic part $\dot{u}_i^*$, i.e., $\dot{u}_i = \dot{u}_i^0 + \dot{u}_i^*$. Then, defining strain rate to be $\dot{\varepsilon}_{ij} = (\dot{u}_{i,j} + \dot{u}_{j,i})/2$, we have

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^0 + \dot{\varepsilon}_{ij}^*, \tag{1}$$

where $\dot{\varepsilon}_{ij}^0 = (\dot{u}_{i,j}^0 + \dot{u}_{j,i}^0)/2$ and $\dot{\varepsilon}_{ij}^* = (\dot{u}_{i,j}^* + \dot{u}_{j,i}^*)/2$. It is noted that $\dot{u}_{i,j}^0$ and $\dot{\varepsilon}_{ij}^0$ are uniform in B , whereas $\dot{u}_{i,j}^*$ and $\dot{\varepsilon}_{ij}^*$ are *kY*-periodic. We assume that each constituent of B has a hypoelastic constitutive relation

$$\dot{\pi}_{ji} = l_{ijkl}\dot{u}_{k,l}, \tag{2}$$

where $\dot{\pi}_{ji}$ denotes the nominal stress rate in the updated Lagrangian formulation, and l_{ijkl} represents microscopic stiffness and satisfies $l_{ijkl} = l_{klij}$. It may be convenient to assume alternatively

$$\dot{s}_{ij} = c_{ijkl}\dot{\varepsilon}_{kl}, \tag{3}$$

where $\dot{s}_{ij}$ indicates Truesdell's stress rate, and $c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$. The two stress rates are related with the material time derivative of Cauchy's stress σ_{ij} as follows:

$$\dot{\pi}_{ji} = \dot{s}_{ij} + \dot{u}_{i,k}\sigma_{jk} = \dot{\sigma}_{ij} + \sigma_{ij}\dot{u}_{k,k} - \sigma_{ik}\dot{u}_{j,k}. \tag{4}$$

Substituting Eqs. (2) and (3) into Eq. (4), we see that $l_{ijkl} = c_{ijkl} + \delta_{ik}\sigma_{jl}$. Here and from now on, δ_{ij} will indicate Kronecker's delta.

We introduce a volume average in *kY* in the current configuration of B ,

$$\langle \# \rangle = |kY|^{-1}\int_{kY} \# \, dY, \tag{5}$$

where $|kY|$ is the current volume of *kY*. Then, using $\dot{u}_i = \dot{u}_i^0 + \dot{u}_i^*$ and the *kY*-periodicity of $\dot{u}_i^*$, we can show that $\langle \dot{u}_{i,j} \rangle = \dot{u}_{i,j}^0$ and $\langle \dot{\varepsilon}_{ij} \rangle = \dot{\varepsilon}_{ij}^0$. Moreover, we can demonstrate that the microscopic relation (4) is transformed to the same form of macroscopic relation

$$\dot{\Pi}_{ji} = \dot{S}_{ij} + \dot{u}_{i,k}^0\Sigma_{jk} = \dot{\Sigma}_{ij} + \Sigma_{ij}\dot{u}_{k,k}^0 - \Sigma_{ik}\dot{u}_{j,k}^0, \tag{6}$$

where $\dot{\Pi}_{ji} = \langle \dot{\pi}_{ji} \rangle$, $\dot{S}_{ij} = \langle \dot{s}_{ij} \rangle$, and $\Sigma_{ij} = \langle \sigma_{ij} \rangle$.

Transforming an equilibrium equation $\dot{\pi}_{ji,j}=0$ into a weak form, and using the *kY*-peridicity of $\dot{\pi}_{ji}$ and $\dot{u}_i$, we can show a virtual work equation

$$\langle \dot{\pi}_{ji}\delta\dot{u}_{i,j}\rangle = \dot{\Pi}_{ji}\delta\dot{u}^0_{i,j}\,, \tag{7}$$

where $\delta\dot{u}_i$ is any variation of $\dot{u}_i$, and $\delta\dot{u}^0_{i,j}=\langle\delta\dot{u}_{i,j}\rangle$. It is noted that if Eq. (7) holds for any $\delta\dot{u}_i$, $\dot{\pi}_{ji}$ is shown to satisfy $\dot{\pi}_{ji,j}=0$ and $\langle\dot{\pi}_{ji}\rangle=\dot{\Pi}_{ji}$.

Substituting Eq. (2) into Eq. (7), and employing Eq. (5) and $\langle\sigma_{jl}\delta\dot{u}^*_{k,j}\rangle=0$, we can derive micro/macro linking equations

$$\dot{\Pi}_{ji}=\left\langle l_{ijkl}(\dot{u}^0_{k,l}+\dot{u}^*_{k,l})\right\rangle,\; \left\langle l_{ijpq}\dot{u}^*_{p,q}\delta\dot{u}^*_{i,j}\right\rangle = -\dot{\varepsilon}^0_{kl}\left\langle c_{ijkl}\delta\dot{u}^*_{i,j}\right\rangle, \tag{8, 9}$$

where $\delta\dot{u}^*_i$ is any velocity field satisfying the *kY*-periodicity. Eq. (8) is the volume average of the microscopic constitutive relation (2) and thus can be regarded as a macroscopic constitutive relation. Eq. (9) is the boundary value problem to find the current field of $\dot{u}^*_i$ in *kY*, $\dot{u}^*_i(\boldsymbol{x},t)$, induced by $\dot{u}^0_{k,l}(t)$.

In Eq. (9), $\dot{u}^*_i$ is linearly related with $\dot{\varepsilon}^0_{kl}$, so that Eq. (9) has a fundamental solution

$$\dot{u}^*_i=\chi_i^{kl}\dot{\varepsilon}^0_{kl}\,. \tag{10}$$

The function χ_i^{kl} in Eq. (10) is determined by solving a boundary value problem

$$\left\langle l_{ijpq}\chi^{kl}_{p,q}\delta\dot{u}^*_{i,j}\right\rangle = -\left\langle c_{ijkl}\delta\dot{u}^*_{i,j}\right\rangle, \tag{11}$$

where χ_i^{kl} is required to be *kY*-periodic. Since $c_{ijkl}=c_{ijlk}$, Eq. (11) gives $\chi_i^{kl}=\chi_i^{lk}$.

Solution (10) provides the macroscopic constitutive relation (8) with the form

$$\dot{\Pi}_{ji}(t)=L^H_{ijkl}(t)\dot{u}^0_{k,l}(t),\; L^H_{ijkl}(t)=\left\langle l_{ijkl}+l_{ijpq}\chi^{kl}_{p,q}\right\rangle. \tag{12, 13}$$

Substituting Eq. (12) into Eqs. (6) and (13), and using Eq. (4) and $\langle\sigma_{jq}\chi^{kl}_{p,q}\rangle=0$, we have

$$\dot{S}_{ij}=C^H_{ijkl}\dot{\varepsilon}^0_{kl}\,,\; C^H_{ijkl}=\left\langle c_{ijkl}+c_{ijpq}\chi^{kl}_{p,q}\right\rangle,\; L^H_{ijkl}=C^H_{ijkl}+\delta_{ik}\Sigma_{jl}\,. \tag{14, 15, 16}$$

We can show that $L^H_{ijkl}=L^H_{klij}$ and $C^H_{ijkl}=C^H_{jikl}=C^H_{ijlk}=C^H_{klij}$. It is emphasized that L^H_{ijkl} is responsible for macroscopic instability. Macroscopic instability occurs, if homogenized incremental stiffness L^H_{ijkl} has a loss of ellipticity. We have examined the onset of macroscopic instability by introducing the acoustic tensor [e.g., 9]. This tensor is defined as $A_{ik}=L^H_{ijkl}n^0_j n^0_l$, where n^0_i indicates the unit normal to a macroscopic surface. Then, the instant we find a direction g^0_k satisfying $A_{ik}g^0_k=0$, instability occurs with respect to macroscopic velocity gradient $g^0_k n^0_l$. Thus, the condition of macroscopic instability is expressed as

$$\det\boldsymbol{A}=0\,. \tag{17}$$

At the onset of microscopic bifurcation, fundamental solution (10) becomes a particular solution, as Eq. (9) comes to have the eigensolution(s) $\phi_i^{(r)}$ determined by

$$\left\langle l_{ijpq}\phi^{(r)}_{p,q}\delta\dot{u}^*_{i,j}\right\rangle=0\,,\; r=1,2,\ldots,m, \tag{18}$$

where *m* denotes the degree of multiplicity. Eq. (9) thus has a bifurcated solution

$$\dot{u}^*_i=\kappa^{(0)}\chi_i^{kl}\dot{\varepsilon}^0_{kl}+\sum^{m}\kappa^{(r)}\phi_i^{(r)}\,, \tag{19}$$

where $\kappa^{(r)}$ $(r=0,1,\ldots,m)$ are constants. We can show that $\phi_i^{(r)}$ satisfies orthogonality

$$\left\langle l_{ijkl}\phi^{(r)}_{i,j}\right\rangle=\left\langle c_{ijkl}\phi^{(r)}_{i,j}\right\rangle=0\,,\; r=1,2,\ldots,m. \tag{20}$$

Results of Analysis

The elastic square honeycombs considered are illustrated in Fig. 1. They are subject to in-plane uniaxial compression. The cell walls of honeycombs have a thickness h and a length l. The elastic properties are Young's modulus E and poison ratio ν .The dashed lines in Fig. 1 indicate examples of the periodic units assumed in this study. They consist of $N \times N$ cells, and this kind of periodic units will be denoted as $Y_{N\times N}$ hereafter.

The analysis of microscopic bifurcation was done, first, by assuming a small periodic unit $Y_{2\times2}$ in the case of $h/l = 0.059$. Microscopic bifurcation then occurred at the point tagged 2×2 on the macroscopic stress-strain relation in Fig. 2, where short wave buckling stress Σ_{SWB} [6,10] is nondimensional parameter. This microscopic bifurcation was simple, i.e., $m = 1$, so that Eq. (18) had an eigensolution, which was uniquely determined. The eigenmode of $Y_{2\times2}$ determined is illustrated in Fig. 3(a).

The analysis was carried further by assuming periodic units $Y_{4\times4}$, $Y_{8\times8}$ and $Y_{16\times16}$. Microscopic bifurcation, then, occurred at smaller compressive stresses than Σ_{SWB}, but

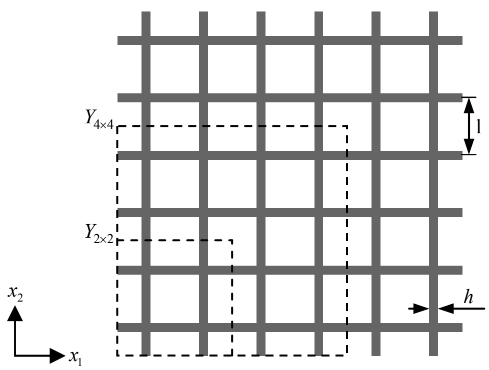

Fig. 1. Periodic units of square honeycombs, and coordinates for analysis.

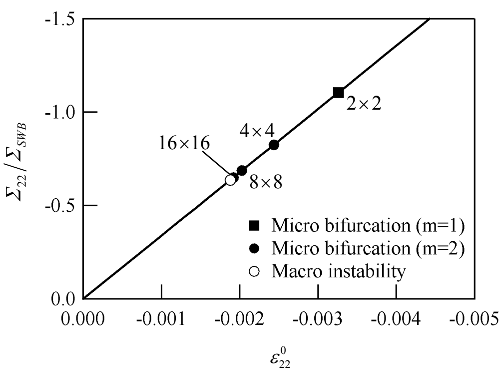

Fig. 2. First microscopic bifurcation points based on $Y_{N\times N}$ ($N = 2, 4, 8, 16$), and macroscopic instability point.

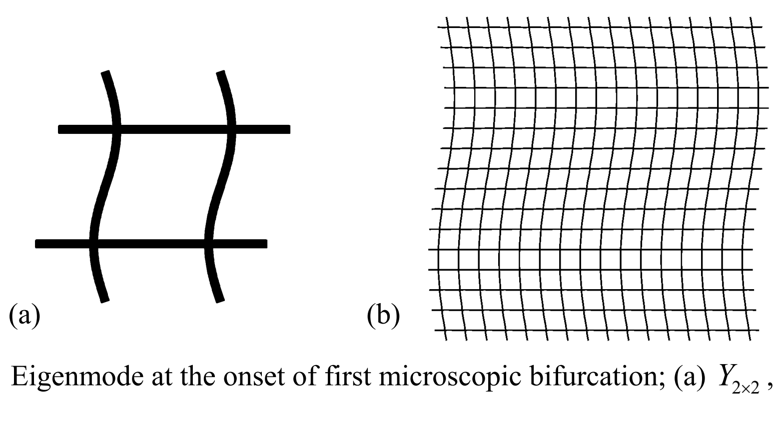

Fig. 3. Eigenmode at the onset of first microscopic bifurcation; (a) $Y_{2\times2}$, (b) $Y_{16\times16}$.

$Y_{8\times8}$ and $Y_{16\times16}$ gave nearly the same bifurcation points (Fig. 2). The microscopic bifurcation thus depended on the number of cells per periodic length. Moreover, the multiplicity was found to be two, i.e., $m=2$, at the bifurcation points by $Y_{4\times4}$, $Y_{8\times8}$ and $Y_{16\times16}$, so that $\phi_i = \kappa^{(1)}\phi_i^{(1)} + \kappa^{(2)}\phi_i^{(2)}$ for these periodic units. This multiplicity was attributable to the two degrees of freedom of long-wave bifurcation, i.e., phase shift and wave form, as was shown for hexagonal honeycombs in the plastic range [5]. We thus found that the eigenmodes were of the shear type, as seen from that of $Y_{16\times16}$ illustrated in Fig. 3(b). These results allowed us to say that the shear type of long-wave bifurcation occurred when $Y_{4\times4}$, $Y_{8\times8}$ and $Y_{16\times16}$ were assumed.

Macroscopic instability condition (17) was examined. Consequently, macroscopic instability occurs when n_i^0 is parallel with the x_2-direction. In this case, Σ_{22} (<0) satisfies the following condition:

$$C_{1212}^H + \Sigma_{22} = 0, \tag{21}$$

i.e., $L_{1212}^H = 0$. Then, we find that g_i^0 is parallel with the x_1-direction. Therefore, we can say that the shear type of macroscopic instability occurs in elastic square honeycombs subject to in-plane uniaxial compression. This result is in accordance with the long-wave bifurcation eigenmode shown in Fig. 3(b).

Let us suppose that the macroscopic instability of elastic square honeycombs occurs at small, macroscopic strains. The variation in C_{1212}^H is, then, negligible before the onset of macroscopic instability, so that we have $C_{1212}^H \approx G_{12}^H$, where G_{12}^H indicates the homogenized in-plane shear rigidity of elastic square honeycombs at small strains. If the beam theory is applied to cell wall deflection [10], and if the flexural rigidity of plates is taken into account for the deflection, G_{12}^H is represented as

$$G_{12}^H = \frac{E}{2(1-\nu^2)}\left(\frac{h}{l}\right)^3 . \tag{22}$$

Then, by substituting Eq. (22) into Eq. (21), the onset stress of macroscopic instability,

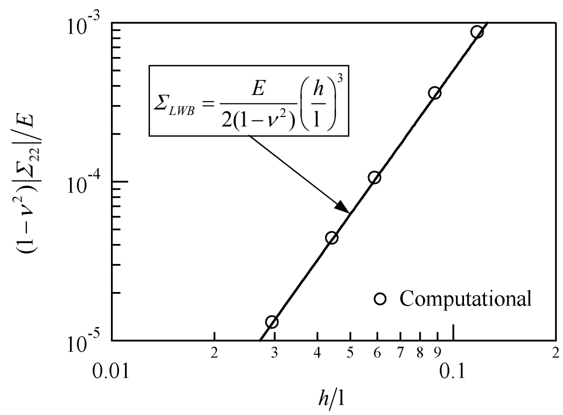

Fig. 4. Comparison of the buckling stress by Eq. (23) and the onset stress of macroscopic instability in finite element computation.

i.e., the lowest stress of long-wave buckling, of elastic square honeycombs subject to in-plane uniaxial compression is evaluated to be

$$\Sigma_{LWB} = \frac{E}{2(1-\nu^2)}\left(\frac{h}{l}\right)^3 . \tag{23}$$

Figure 4 compares the buckling stress by Eq. (23) and the onset stress of macroscopic instability obtained in the present computation.

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