Point Load Generated Surface Waves in a Transversely Isotropic Half-Space using Elastodynamic Reciprocity

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Summary

Surface waves generated by sub-surface time-harmonic point-loads in a transversely isotropic elastic half-space are considered. The numerical results computed are compared with the Green's function solutions obtained previously.

Introduction

Point-load problems are usually solved by an application of integral transform techniques, and an evaluation of the related inverse transforms by contour integration and residue calculus. However, the resulting integrals in general are not easy to evaluate analytically. Recently elastodynamic reciprocity has been used to study time-harmonic point-loads in an isotropic elastic half-space [1, 2].

This paper extends the use of elastodynamic reciprocity in [1, 2] to a transversely isotropic half-space, using the same techniques. However, as mentioned in [1, 2] the calculation does not include a consideration of the body waves generated by the point-loads. The numerical results has been compared with the Green's function solutions [3], and when the material is reduced to the isotropic elastic case, the solutions of [1, 2] are recovered.

Basic Equations

The homogeneous transversely isotropic elastic half-space and the Cartesian coordinate system $Ox_1x_2x_3$ are shown in Fig. 1. The x_1x_2 - plane coincides with the surface of the half-space and the x_3-axis is perpendicular to the plane of isotropy. In Fig. 1, the time harmonic point-load F has a vertical component P and a horizontal component Q.

Considering the formulation of [4], solutions for displacement components are taken as,

$$u_\alpha(\mathbf{x},t)=\frac{1}{k}V(x_3)\frac{\partial\varphi(x_1,x_2)}{\partial x_\alpha}e^{i\omega t},\quad u_3(\mathbf{x},t)=W(x_3)\varphi(x_1,x_2)e^{i\omega t},\tag{1}$$

where $\alpha=1,2$ and $k=\omega/c$. In the following analysis, the factor $e^{i\omega t}$ is omitted, and Greek indices refer to x_1, x_2. Solutions of the form given by Eq. (1) satisfy

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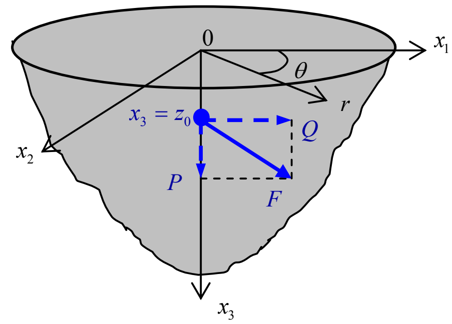


Fig. 1. Transversely isotropic elastic half-space subjected to a subsurface time-harmonic point-load.

the elastodynamic equations of motion, if the dimensionless function $\varphi(x_1,x_2)$ is taken as the solution of

$$\varphi(x_1,x_2)_{,\alpha\alpha} + k^2\varphi(x_1,x_2) = 0, \tag{2}$$

where repeated indices indicate summation, and $V(x_3)$ and $W(x_3)$ are solutions of the following system of ordinary differential equations,

$$\begin{aligned} &C_{44}V''(x_3) - (k^2 C_{11} - \rho\omega^2)V(x_3) + k(C_{13}+C_{44})W'(x_3) = 0, \\ &C_{33}W''(x_3) - (k^2 C_{44} - \rho\omega^2)W(x_3) - k(C_{13}+C_{44})V'(x_3) = 0. \end{aligned} \tag{3}$$

Here, C_{ij} are the elastic constants of transversely isotropic medium, ρ is the mass density and prime indicates d/dx_3 .

For the elastic half-space $x_3 \geq 0$, solutions of Eq. (3) that decay exponentially with the depth can be written as

$$V(x_3) = A_0 e^{-ks_1x_3} + B_0 e^{-ks_2x_3}, \quad W(x_3) = m_1 A_0 e^{-ks_1x_3} + m_2 B_0 e^{-ks_2x_3}, \tag{4}$$

where A_0, B_0 are constants of length dimension and

$$\begin{aligned} &k = \omega / c, \quad m_\alpha = [\rho c^2 - C_{11} + s_\alpha^2 C_{44}]/[s_\alpha(C_{13}+C_{44})], \\ &s_1^2, s_2^2 = [-R_3 \pm (R_3^2 - 4C_{33}C_{44}R_1R_2)^{\frac{1}{2}}]/[2C_{33}C_{44}], \end{aligned} \tag{5}$$

in which

$$R_1 = \rho c^2 - C_{11}, \; R_2 = \rho c^2 - C_{44}, \; R_3 = (C_{13}+C_{44})^2 + R_1C_{33} + R_2C_{44}. \tag{6}$$

From Eq. (4) the solutions which satisfy the traction free conditions at $x_3 = 0$ plane, can be written as,

$$V(x_3) = A_0[e^{-k_R s_1 x_3} - \tfrac{s_1 - m_1}{s_2 - m_2} e^{-k_R s_2 x_3}], \quad W(x_3) = A_0[m_1 e^{-k_R s_1 x_3} - m_2 \tfrac{s_1 - m_1}{s_2 - m_2} e^{-k_R s_2 x_3}], \quad (7)$$

where c_R is the surface wave speed calculated from

$$\rho c_R^2 R_1 - s_1 s_2 [C_{33} R_1 + C_{13}^2] = 0. \quad (8)$$

Surface Wave Motion Generated by P

Equation (1) can be rewritten in the cylindrical coordinate system (r,θ,z) as

$$u_r(r,z) = \frac{1}{k_R} V(z) \frac{\partial \varphi(r,\theta)}{\partial r}, \qquad u_z(r,z) = W(z)\varphi(r,\theta), \quad (9)$$

and Eq. (2) becomes

$$\frac{\partial^2 \varphi(r)}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi(r)}{\partial r} + k_R^2 \varphi(r) = 0. \quad (10)$$

For this axially symmetric case, the relevant solution of Eq. (10) for an outgoing wave is then a Hankel function,

$$\varphi(r) = H_0^{(2)}(k_R r). \quad (11)$$

Then Eq. (9) is simplified to

$$u_r(r,z) = -A_0 \bar{V}(z) H_1^{(2)}(k_R r), \qquad u_z(r,z) = A_0 \bar{W}(z) H_0^{(2)}(k_R r), \quad (12)$$

where $A_0\bar{V}(z) = V(z)$ and $A_0\bar{W}(z) = W(z)$. Expressions for the corresponding stresses can be obtained from Hooke's law as

$$\begin{aligned}
\sigma_{rz}(r,z) &= -A_0 C_{44}[\bar{V}'(z) + k_R \bar{W}(z)] H_1^{(2)}(k_R r), \\
\sigma_{zz}(r,z) &= -A_0 [C_{13} k_R \bar{V}(z) - C_{33}\bar{W}'(z)] H_0^{(2)}(k_R r), \qquad (13) \\
\sigma_{rr}(r,z) &= -A_0 \left\{ [C_{11} k_R \bar{V}(z) - C_{13}\bar{W}'(z)] H_0^{(2)}(k_R r) - 2C_{66}\bar{V}(z) H_1^{(2)}(k_R r)/r \right\}.
\end{aligned}$$

Use of Elastodynamic Reciprocity

Considering two distinct time-harmonic states of the same frequency denoted by the superscripts *A* and *B*, the reciprocity relation is given by

$$\int_V (f_i^A u_i^B - f_i^B u_i^A)\, dV = \int_S (u_i^A \sigma_{ij}^B - u_i^B \sigma_{ij}^A)\, n_j \, dS, \quad (14)$$

where f_i^A, f_i^B are body forces, $\sigma_{ij}^A, \sigma_{ij}^B$ are stresses and u_i^A, u_i^B are displacements, and n_j are the components of the outward normal. Here repeated indices imply summation. For V, the regions are defined by $0 \le r \le a$, $0 \le z < \infty$,

$0 \le \theta \le 2\pi$. For state *A*, actual displacements and stresses are given by Eqs. (12) and (13), while for state *B*, dummy solutions consisting of the sum of an outgoing wave and a converging wave are selected as follows [2],

$$\begin{aligned} u_r^B(r,z) &= -\bar{V}(z)[H_1^{(1)}(k_R r) + H_1^{(2)}(k_R r)]/2, \\ u_z^B(r,z) &= \bar{W}(z)[H_0^{(1)}(k_R r) + H_0^{(2)}(k_R r)]/2. \end{aligned} \tag{15}$$

Since the displacements defined by Eq. (15) is bounded at $r = 0$, it can be verified that the left-hand side of Eq. (14) becomes

$$P u_z^B(0, z_0) = P\bar{W}(z_0), \tag{16}$$

where the force P is applied at $z = z_0$. Hence, Eq. (14) is expressed as follows

$$P\bar{W}(z_0) = a\int_0^{2\pi}\int_0^{\infty}\Big[\left(u_r^A\sigma_{rr}^B - u_r^B\sigma_{rr}^A\right) + \left(u_z^A\sigma_{rz}^B - u_z^B\sigma_{rz}^A\right) + \left(u_\theta^A\sigma_{r\theta}^B - u_\theta^B\sigma_{r\theta}^A\right)\Big]d\theta\, dz. \tag{17}$$

Substitution of the corresponding expressions for displacements and stresses for state *A* and *B* into Eq. (17) yields

$$P\bar{W}(z_0) = -\pi a A_0 I\left[H_0^{(2)}(k_R a)H_1^{(1)}(k_R a) - H_1^{(2)}(k_R a)H_0^{(1)}(k_R a)\right], \tag{18}$$

where I is defined as

$$I = \int_0^{\infty}\left\{[-C_{11}k_R\bar{V}(z) + C_{13}\bar{W}'(z)]\bar{V}(z) - C_{44}[\bar{V}'(z) + k_R\bar{W}(z)]\bar{W}(z)\right\}dz, \tag{19}$$

and Eq. (18) can be rewritten as

$$P\bar{W}(z_0) = -4A_0 i\, I / k_R. \tag{20}$$

By using Eqs. (12) and (20), the displacement components at position (r,z) are obtained as

$$\begin{aligned} u_r(r,z) &= -i\,k_R P\bar{W}(z_0)\bar{V}(z)H_1^{(2)}(k_R r)/(4I), \\ u_z(r,z) &= i\,k_R P\bar{W}(z_0)\,\bar{W}(z)H_0^{(2)}(k_R r)/(4I). \end{aligned} \tag{21}$$

In a similar manner, surface wave displacements due to the horizontal component Q are obtained as

$$\begin{aligned} u_r(r,\theta,z) &= ik_R Q\bar{V}(z_0)\bar{V}(z)[H_0^{(2)}(k_R r) - (k_R r)^{-1}H_1^{(2)}(k_R r)]\cos\theta/(4I), \\ u_z(r,\theta,z) &= ik_R Q\bar{V}(z_0)\bar{W}(z)H_1^{(2)}(k_R r)\cos\theta/(4I), \\ u_\theta(r,\theta,z) &= -iQ\bar{V}(z_0)\bar{V}(z)H_1^{(2)}(k_R r)\sin\theta/(4rI). \end{aligned} \tag{22}$$

Numerical Results

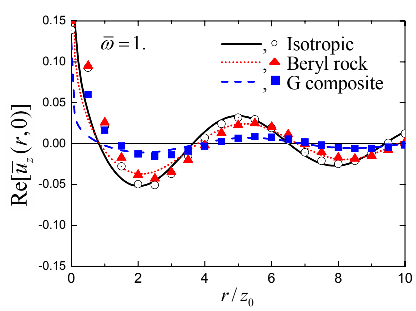
The elastic constants of transversely isotropic and isotropic materials used in [3] are given in Table 1.

Table 1 Elastic constants of materials.

Materials	$\bar{C}_{11}$	$\bar{C}_{12}$	$\bar{C}_{13}$	$\bar{C}_{33}$	$C_{44}\left(\times 10^{4}\ N/mm^{2}\right)$
Isotropic ($\nu = 0.25$)	3.0	1.0	1.0	3.0	1.0
Beryl rock	4.13	1.47	1.01	3.62	1.0
Graphite/epoxy composite	2.024	0.683	0.073	21.17	0.41

note: $\bar{C}_{ij} = C_{ij}/C_{44}$

The results of the present study for a vertical load P are compared with the Green's function solutions of [3]. For a prescribed frequency $\bar{\omega} = \omega z_0 \sqrt{\rho / C_{44}}$ $= 1.0$, the non-dimensional vertical displacement along the free surface, $\bar{u}_z(r,0) = u_z(r,0) C_{44} z_0 / P$ and the non-dimensional normal stress at $z = z_0$, $\bar{\sigma}_{zz}(r, z_0) = \sigma_{zz}(r, z_0) z_0^2 / P$ are plotted in Figs. 2 and 3, respectively. It can be seen in Figs. 2 and 3 that when $r / z_0 \to 0$ the real parts of the displacements and stresses of the present study, are infinite while the imaginary parts are finite. Since the point load P is applied below the surface at $z = z_0$, the displacement should be bounded at the free surface when $r / z_0 \to 0$. But when $r / z_0 \to 0$ the real part of the displacement $\mathrm{Re}[\bar{u}_z(r,0)]$ are unbounded in the present study (Fig. 2). This may be due to the fact that only surface waves are considered in this solution. When r / z_0 ? 0 the present solution agrees well with Ref. [3].



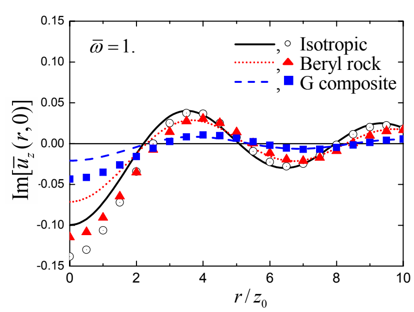


Fig. 2. Displacement in z-direction; lines - present study, symbols - Ref. [3].

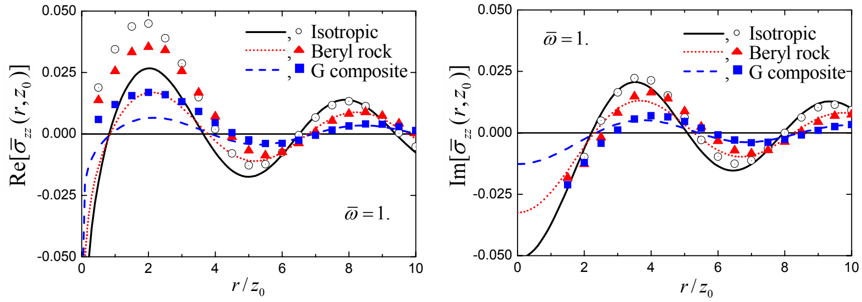


Fig. 3. Normal stress $\bar{\sigma}_{zz}(r, z_0)$; lines - present study, symbols - Ref. [3].

Conclusions

Generally, it is well known that the surface wave dominates wave fields far from the applied load. Figures 2 and 3 show that as the radial distance increases, the present solution agrees well with the Green's function solution [3]. Therefore the accuracy of expressions for surface wave displacements is confirmed.

References

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