On Fracture Mode Mixity Measures in Interface Cracks. An Application to BEM Analysis of Fibre/Matrix Debonding in Composites

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Summary

Linear elastic open model of interface cracks between two isotropic materials is considered in the plane strain state. A new simple equation relating the two measures of fracture mode mixity: the *local phase angle* ψ_K based on the Stress Intensity Factor (SIF) and the *energetic angle* ψ_G based on the Energy Release Rate (ERR), is presented and analysed. An oscillatory behaviour of the components of ERR with unexpected slightly negative values for certain intervals of the finite virtual crack extension is observed and studied. Applying the equation obtained it is now possible to work with both measures in as near–equivalent a manner as possible. This is illustrated by a Boundary Element Method (BEM) analysis of the fibre/matrix debonding growth in a glass fibre reinforced composite material subjected to a transversal load.

Introduction

Oscillatory behaviour is an inherent feature of a linear elastic solution of the *open model* of interface cracks [1] for Dundurs parameter $\beta \neq 0$. Stresses and displacements start to oscillate when the crack tip is approached. A consequence of these oscillations is that this solution predicts an infinite number of zones, where the crack faces interpenetrate. As was shown in [2], this oscillatory behaviour is avoided assuming a contact zone adjacent to the crack tip in the *contact model* of interface cracks. However, in practice the region of these oscillations is frequently physically non-relevant (due to its atomic or subatomic size). The concept of *small-scale contact* (SSC), introduced in [3] to characterize such a situation, provides a theoretical base for applications of the open model to interface crack growth predictions in many problems [4].

An interface crack, assuming a non-vanishing β, is growing inherently in a mixed mode independently of the load applied, with both normal and shear stresses acting at the interface ahead of the crack tip. In order to measure *fracture mode mixity*, the SIF and ERR based approaches have been traditionally used.

Whereas the SIF based measure of mode mixity ψ_K is easy to identify from the singular asymptotic term of the linear elastic solution of the open model of interface cracks [3,4], the ERR based mode mixity measure ψ_G is obtained through complex integrals, which in some way have obscured the relation between these two mode mixity measures. Although several fundamental works were previously published with regards to this relation [5,6,7,8,9,10], this question has only recently been clarified in [12], some results of this work being shortly introduced in the present article. Finally, these theoretical results are applied in a BEM analysis of a fibre/matrix debonding subjected to a transversal load.

Near-Tip Solution of the Open Model

Let the local cartesian system (x,y) and polar coordinate system (r,θ) be defined at the interface crack tip as shown in Fig. 1. Let G_k denote the shear modulus and ν_k the Poisson ratio of material $k = 1,2$. Then the Dundurs bi-material mismatch parameter β is given as:

$$\beta = \frac{G_1(\kappa_2-1)-G_2(\kappa_1-1)}{G_1(\kappa_2+1)+G_2(\kappa_1+1)}, \quad |\beta| \leq 0.5, \qquad \kappa_k = 3-4\nu_k. \tag{1}$$

The oscillation index of a bi-material is expressed in terms of β as:

$$\varepsilon = \frac{1}{2\pi}\ln\frac{1-\beta}{1+\beta}, \quad |\varepsilon| \leq \frac{\ln 3}{2\pi} \doteq 0.175. \tag{2}$$

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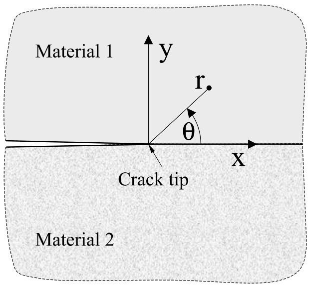

Figure 1: Local coordinate systems at the interface crack tip.

The near-tip tractions along the bonded interface part ($\theta = 0$) are expressed as

$$\sigma_{yy}(r,0) + i\sigma_{xy}(r,0) = \frac{K(r/l)^{i\varepsilon}}{\sqrt{2\pi r}}, \quad \text{for } r \to 0, \tag{3}$$

where r is the distance from the tip, $i = \sqrt{-1}$, and $K = K_1 + iK_2$ is the complex SIF, associated to a reference length l according to [3].

The near-tip relative displacements across the crack $\Delta u_i(r) = u_i(r,\pi) - u_i(r,-\pi)$ are expressed by:

$$\Delta u_y(r) + i\Delta u_x(r) = \frac{8}{1+2i\varepsilon} \frac{K(r/l)^{i\varepsilon}}{\cosh(\pi\varepsilon)E^*} \sqrt{\frac{r}{2\pi}}, \quad \text{for } r \to 0, \quad \frac{1}{E^*} = \frac{1}{2}\left(\frac{1}{E_1'} + \frac{1}{E_2'}\right), \tag{4}$$

$E_k' = E_k/(1-\nu_k^2)$, E_k being the Young elasticity modulus of material k.

SIF Based Fracture Mode Mixity

The *fracture mode mixity measure based on the SIF* is given by the so-called *local phase angle* ψ_K defined by $K = |K|e^{i\psi_K}$ or equivalently as:

$$\psi_K = \arg K = \arg\{\sigma_{yy}(l,0) + i\sigma_{xy}(l,0)\} = \arg\{\Delta u_y(l) + i\Delta u_x(l)\} + \arctan(2\varepsilon), \tag{5}$$

where arg is the argument function and a sufficiently small l is considered. $|K|$ is independent of l. According to (5), ψ_K is an l-dependent measure of the fracture mode mixity, and K is rotating when $l \to 0$. In particular, (3) implies that ψ_K and $\tilde{\psi}_K$ associated to two different reference lengths l and $\tilde{l}$ are related by [3]:

$$\tilde{\psi}_K = \psi_K + \varepsilon \ln(\tilde{l}/l). \tag{6}$$

ERR Based Fracture Mode Mixity

Application of the virtual crack closure method to an interface crack, considering a small but finite length Δa of a virtual crack extension along the interface, gives ERR associated to this crack extent [5,6,7]:

$$G^{\text{int}}(\Delta a) = G_I^{\text{int}}(\Delta a) + G_{II}^{\text{int}}(\Delta a), \tag{7}$$

$$G_I^{\text{int}}(\Delta a) = \frac{1}{2\Delta a}\int_0^{\Delta a} \sigma_{yy}(r,0)\Delta u_y(\Delta a - r)\mathrm{d}r, \quad G_{II}^{\text{int}}(\Delta a) = \frac{1}{2\Delta a}\int_0^{\Delta a} \sigma_{xy}(r,0)\Delta u_x(\Delta a - r)\mathrm{d}r. \tag{8}$$

The total ERR G^{int}, associated to an infinitesimal virtual crack extension, in terms of K writes as [11]:

$$G^{\text{int}} = \lim_{\Delta a \to 0} G^{\text{int}}(\Delta a) = \frac{|K|^2}{\cosh^2(\pi\varepsilon)E^*}. \tag{9}$$

Thus, G^{int} depends only on $|K|$ and not on ψ_K. The minimum value of ERR at a prescribed ψ_K that originates an interface decohesion is called *interface toughness* at this fracture mode mixity, $G_c^{\text{int}}(\psi_K)$.

Due to the oscillatory character of the near-tip elastic field, $G_{I,II}^{\text{int}}(\Delta a)$ oscillate as well and consequently their limits do not exist as $\Delta a \to 0$. This oscillatory behaviour was studied by several authors [5,6,7]. The following new explicit expressions of the individual components of the energy release rate associated to Δa, considering only singular terms in (3) and (4), has recently been deduced in [12] developing a result in [7]:

$$G_{I,II}^{\text{int}}(\Delta a) = 0.5G^{\text{int}}\left[1 \pm F(\varepsilon)\cos\{2(\psi_K + \psi_0(\Delta a/l, \varepsilon))\}\right], \tag{10}$$

where the amplitude function $F(\varepsilon)$ and the phase shift angle $\psi_0(\Delta a/l, \varepsilon)$ are expressed as

$$F(\varepsilon) = \sqrt{\frac{\sinh(2\pi\varepsilon)}{2\pi\varepsilon(1+4\varepsilon^2)}} = 1 + \left(\frac{\pi^2}{3} - 2\right)\varepsilon^2 + O(\varepsilon^4), \tag{11}$$

$$\psi_0(\Delta a/l, \varepsilon) = \varepsilon \ln(\Delta a/2l) + \varphi(\varepsilon) - 0.5\arctan(2\varepsilon) = \varepsilon \ln(\Delta a/4el) + (\zeta(3) + 4/3)\varepsilon^3 + O(\varepsilon^5), \tag{12}$$

$$\varphi(\varepsilon) = 0.5\arg\left[\Gamma(0.5 + i\varepsilon)/\Gamma(1 + i\varepsilon)\right] = -\varepsilon \ln 2 + \zeta(3)\varepsilon^3 + O(\varepsilon^5), \tag{13}$$

$\Gamma(\cdot)$ and $\zeta(\cdot)$ respectively denoting the gamma and Riemann zeta functions, $\zeta(3) \doteq 1.2020569$ being Apéry's constant and $e \doteq 2.718$ being the base of the natural logarithm.

A consequence of oscillations in $G_{I,II}^{\text{int}}(\Delta a)$ is that the *energetic angle* ψ_G defined as:

$$\tan^2\psi_G = \frac{G_{II}(\Delta a)}{G_I(\Delta a)}, \qquad 0 \le \psi_G \le \frac{\pi}{2}, \tag{14}$$

depends on Δa. The fact that, for small ε, ψ_G is a gentle function of Δa inside of a physically relevant interval of Δa (in a similar way as ψ_K is a gentle function of l), is used by some authors as a justification for application of this *ERR based fracture mode mixity measure* to predict interface crack behaviour.

A relevant aspect of the oscillatory behaviour of $G_{I,II}^{\text{int}}(\Delta a)$ according to (10) is associated to the somewhat unexpected property of the amplitude function $F(\varepsilon)$, clearly seen from its series expansion in (11): $F(\varepsilon) > 1$ for $\varepsilon \neq 0$. A crucial consequence of this fact is that there is an infinite number of intervals of values of Δa, as $\Delta a \to 0$, where one ERR component is slightly negative. The maximum negative value of one ERR component is an increasing function of $|\varepsilon|$ and is less than 2% of $G^{\text{int}}(\Delta a)$. Fig. 2 illustrates this phenomenon for the maximum value of $\varepsilon = 0.175$ in logarithmic and standard scales of $\Delta a/l$. Following (8), the different signs that stresses and associated relative displacements may have at both sides of the interface crack tip are responsible for the negative value of either $G_I(\Delta a)$ or $G_{II}(\Delta a)$.

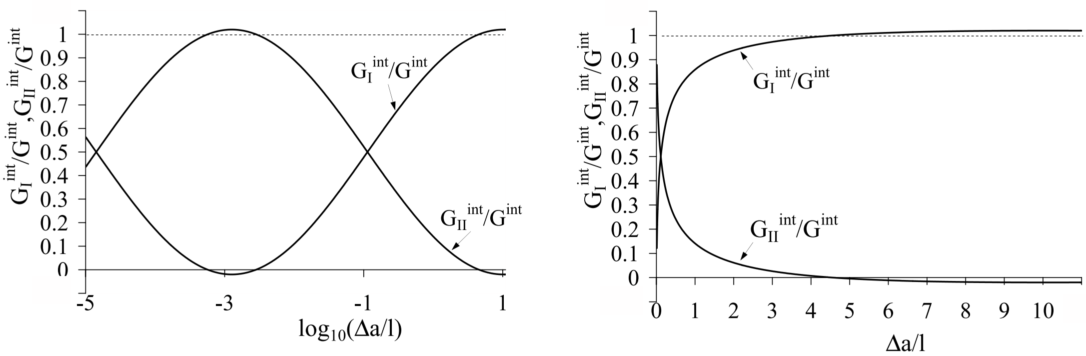

Figure 2: Variation of ERR with virtual crack extension length ($K = 1$, $\varepsilon = 0.175$).

It is useful to observe that $\psi_0(\Delta a/l, \varepsilon) = 0$ when $\Delta a/l = 2\exp[(0.5\arctan(2\varepsilon) - \varphi(\varepsilon))/\varepsilon]$, which gives the following interval for such values of $\Delta a/l$: $10.1169 < \Delta a/l < 10.8731$. Thus, the value of $\Delta a/l$ giving a vanishing shift angle ψ_0 is found to be quite independent of ε.

The following fundamental relation between ψ_K and ψ_G is obtained by substituting (10) into (14):

$$\cos(2\psi_G) = F(\varepsilon)\cos\{2(\psi_K + \psi_0(\Delta a/l, \varepsilon))\}, \quad (15)$$

which implies that

$$\psi'_K = 0.5\arccos\left[F(\varepsilon)^{-1}\cos(2\psi_G)\right], \quad (16)$$
$$\psi_G = 0.5\arccos\left[F(\varepsilon)\cos\{2(\psi_K + \psi_0(\Delta a/l, \varepsilon))\}\right], \quad (17)$$

where $\psi'_K = |\psi_K + \psi_0(\Delta a/l, \varepsilon) + n\pi|$, n being an integer number (usually $n = 0, \pm 1$) giving $0 \le \psi'_K \le \pi/2$. Fig. 3 illustrates the explicit relation obtained between ψ_K and ψ_G. As can be seen, values of ψ'_K are relatively well approximated by values of ψ_G excepting zones where ψ_G is close to 0 or $\pi/2$. This is true in particular for small values of ε. However, a ‘strange’ behaviour can be observed in Fig. 3: for the extremal values of ψ'_K near 0 and $\pi/2$ there are no corresponding real values of ψ_G. In fact, values of ψ_G corresponding to these values of ψ'_K are pure imaginary or complex numbers because in such a situation either $G_I(\Delta a)$ or $G_{II}(\Delta a)$ is negative (see Fig. 2), and consequently, in view of (14), $\tan^2\psi_G$ is negative. For further details of this behaviour and also of other aspects of the relation between ψ_K and ψ_G, see [12].

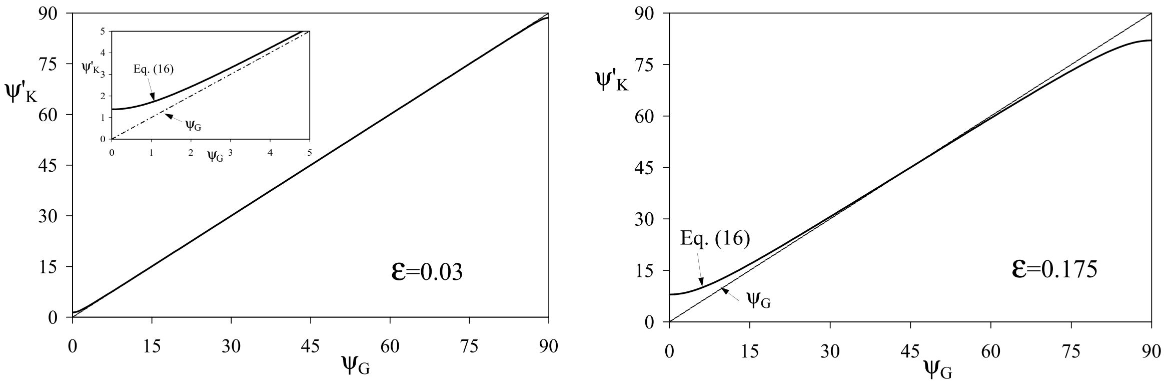

Figure 3: ψ'_K as a function of ψ_G.

BEM Analysis of Fibre/Matrix Debonding

Fibre reinforced composites subjected to loads perpendicular to the direction of the fibres suffer failures known as matrix or inter–fibre failures. These failures typically involve debondings between matrix and fibre, which can be considered as interface cracks. A simple micromechanical model of this kind of damage is studied in this section with the aim to contribute to the understanding of the mechanism of its propagation.

The configuration of a single fibre surrounded by matrix with a partial debonding subjected to the unit far–field tension σ perpendicular to this debonding is represented in Fig. 4. The following properties of the glass (fibre)–epoxy (matrix) bimaterial system are considered: $G^f = 29GPa$, $G^m = 1.05GPa$, $\nu^f = 0.22$, $\nu^m = 0.33$, which in plane strain state yield $\beta = 0.229$ and $\varepsilon = -0.0742$. The glass fibre radius is $r^f = 7.5\mu m$. For more details about this problem see [13,14]. In the BEM model developed, using continuous linear elements [15] with analytical integrations, the possibility to detect a near tip (frictionless) contact zone between crack faces is applied. A strong refinement of the boundary element mesh is used to this aim near the crack tip.

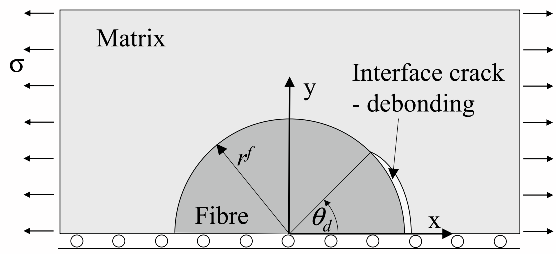

Figure 4: The single fibre model employed.

Although the interface crack between fibre and matrix is curved, it is supposed that locally the open model theory, developed originally for a straight crack, is applicable with a reasonable accuracy.

Values of $G^{\text{int}}(\Delta a)$ and its components, obtained by BEM using (7-8) for Δa corresponding to 0.5°, are shown in Fig. 5a as functions of the debonding angle θ_d. The maximum of $G^{\text{int}}(\Delta a)$ coincides with the maximum of $G_{II}^{\text{int}}(\Delta a)$ at approximately $\theta_d = 60^\circ \sim 70^\circ$, where $G_I^{\text{int}}(\Delta a)$ vanishes. Starting from this debonding angle the near tip contact zone becomes of a physically relevant size and the open model is then not valid for larger debondings. Therefore, the subsequent considerations will be limited to smaller debondings.

Values of $G_{I,II}^{\text{int}}(\Delta a)$ shown in Fig. 5a are used to evaluate $\psi_G(\theta_d)$ using definition (14), see Fig. 5b. Taking the characteristic length l in such a way that $\psi_0(\Delta a/l, \varepsilon) = 0$, (16) is applied to evaluate ψ_K. For comparison purposes $\psi_K(\theta_d)$ is also evaluated using values of stresses ahead of the crack tip and relative displacements between crack faces, both taken at the distance l from the crack tip. It can be observed in Fig. 5b that ψ_G and ψ_K are almost coincident up to $\theta_d \simeq 45^\circ$. Then, as could be expected in view of Fig. 3, ψ_G starts to be larger than ψ_K due to the fact that ψ_G is relatively close to 90°. Fig. 5b shows a very good agreement between ψ_K values obtained from ψ_G and from stress and displacement values up to $\theta_d \simeq 50^\circ$. The major difference at $\theta_d \simeq 60^\circ$ is possibly related to the fact that at this θ_d a physically relevant contact zone appears and the open model solution does not fit well the BEM solution obtained.

Conclusions

The present work contributes to clarify an existing natural duality between SIF and ERR based concepts in the linear elastic open model of Interfacial Fracture Mechanics. The significance of the simple equation relating ψ_K and ψ_G introduced is associated to the fact that these angles are key parameters in the characterization of interfacial fracture toughness G_c^{int} [4], which is given as a function of one particular angle (ψ_K or ψ_G). The present equation makes it possible to transform easily a toughness function of one angle to the toughness function of the other angle. This possibility will be used in a BEM study of the fibre/matrix debonding growth in a composite lamina subjected to a transversal load, which will be presented in a forthcoming work. It is expected that the procedure of ψ_K evaluation from ψ_G through (8), (14) and (16) provides a higher accuracy, due to integrations present in (8), in comparison with an application of point values of either stresses or relative displacements according to (5). A comparison with analytical values of fracture mode mixity will be required to confirm this hypothesis.

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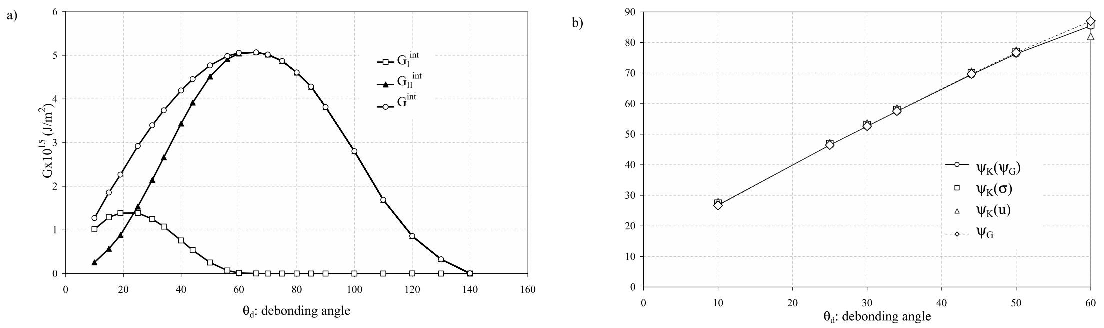

Figure 5: a) $G^{\text{int}}(\Delta a)$ and $G^{\text{int}}_{I,II}(\Delta a)$ as a function of θ_d, b) ψ_G and ψ_K as a function of θ_d.

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