Advances in Computational & Experimental Engineering & Science
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The Novel Concept "Tangential Relaxation"

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Summary

The novel concept "*tangential stress rate relaxation*", abbreviated as "*tangential relaxation*", is proposed in order to predict rigorously the plastic instability phenomena in which the stress rate has a tangential component deviating severely from the proportional loading. Further, the constitutive equation based on this concept is formulated.

Introduction

Traditional plasticity is concerned only with the stress rate component normal to the yield surface but is independent of the tangential component. Thus, it predicts an unrealistically stiff mechanical response in plastic instability phenomena in which the stress rate has a tangential component deviating severely from the proportional loading. In order to improve this defect in the traditional theory, various constitutive models have been proposed so far. Among them only the tangential inelasticity model [1] which incorporates the inelastic strain rate induced by the stress rate component tangential to the subloading surface is applicable to the general loading process, which is regarded as the modification of Rudnicki and Rice's [2] rate form of the J_2-deformation theory by the concept of the subloading surface model [3, 4]. However, it is not derived from the physically rigorous background.

In this article the novel concept "*tangential stress rate relaxation*", abbreviated as "*tangential relaxation*", is proposed in order to predict rigorously the plastic instability phenomena and the constitutive equation based on this concept is formulated.

Outline of the Subloading Surface Model

Let the strain rate $\mathbf{D}$ be additively decomposed into the elastic strain rate $\mathbf{D}^e$ and the inelastic strain rate $\mathbf{D}^p$, i.e.

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p , \tag{1}$$

where $\mathbf{D}^e$ is given by

$$\mathbf{D}^e = \mathbf{E}^{-1} \overset{\circ}{\boldsymbol{\sigma}} . \tag{2}$$

$\boldsymbol{\sigma}$ is the Cauchy stress and $(\,^{\circ}\,)$ indicates the proper corotational rate (see Appendix) and the fourth-order tensor $\mathbf{E}$ is the elastic modulus.

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Proceedings of the 2004 International Conference on Computational & Experimental Engineering & Science
26-29 July, 2004, Madeira, Portugal

The following *subloading surface* is introduced.

$$f(\hat{\boldsymbol{\sigma}}, \mathbf{H}) = RF(H), \tag{3}$$

$$\hat{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \boldsymbol{\alpha}. \tag{4}$$

The scalar H and the second-order tensor $\mathbf{H}$ are the isotropic and the anisotropic hardening variables, respectively, $\boldsymbol{\alpha}$ is the kinematic hardening variable, i.e. the back stress. The function f is assumed to be homogeneous of degree one in the stress $\hat{\boldsymbol{\sigma}}$. R is the ratio of the size of the subloading surface to that of the normal-yield surface and is called the *normal-yield ratio*. Its evolution equation is given as follows:

$$\dot{R} = U\|\mathbf{D}^p\| \text{ for } \mathbf{D}^p \neq \mathbf{0}, \tag{5}$$

where U is a monotonically-decreasing function of the normal-yield ratio R, fulfilling the conditions

$$U = \begin{cases} \infty & \text{for } R = 0, \\ 0 & \text{for } R = 1. \end{cases} \tag{6}$$

Further, assume the flow rule

$$\mathbf{D}^p = \lambda \mathbf{M}, \tag{7}$$

where λ is a positive proportionality factor and $\mathbf{M}$ is the direction of plastic strain rate.

Substitution of Eqs. (5) and (7) into the time-differentiation of Eq. (1) leads to

$$\lambda = \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^p}, \tag{8}$$

where

$$M^p \equiv \mathrm{tr}\Big[\mathbf{N}\Big[\Big(\frac{F'}{F}h + \frac{U}{R}\Big)\hat{\boldsymbol{\sigma}} + \mathbf{a} - \frac{1}{RF}\mathrm{tr}\Big\{\frac{\partial f(\hat{\boldsymbol{\sigma}}, \mathbf{H})}{\partial \mathbf{H}}\mathbf{h}\Big\}\hat{\boldsymbol{\sigma}}\Big]\Big]. \tag{9}$$

$$\mathbf{N} \equiv \frac{\partial f(\hat{\boldsymbol{\sigma}}, \mathbf{H})}{\partial \boldsymbol{\sigma}} \Bigg/ \left\|\frac{\partial f(\hat{\boldsymbol{\sigma}}, \mathbf{H})}{\partial \boldsymbol{\sigma}}\right\| \quad (\|\mathbf{N}\| = 1),\ F' \equiv dF/dH,\ h \equiv \frac{\dot{H}}{\lambda},\ \mathbf{h} \equiv \frac{\overset{\circ}{\mathbf{H}}}{\lambda},\ \mathbf{a} \equiv \frac{\overset{\circ}{\boldsymbol{\alpha}}}{\lambda}. \tag{10}$$

The strain rate is given from Eqs. (1), (2), (7) and (8) as

$$\mathbf{D} = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} + \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^p}\mathbf{M}. \tag{11}$$

Tangential Relaxation

From Eq. (11) one has

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{E}\mathbf{D} - \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^p}\mathbf{E}\mathbf{M}, \tag{12}$$

It is observed in Eq. (12) that the relaxation is induced by the stress rate component normal to the subloading surface. Let it be called the "*normal stress rate relaxation*", abbreviated as "*normal relaxation*". Now, let it be postulated that the relaxation is induced by the stress rate component tangential to the subloading surface, called the tangential stress rate. Let it be called the "*tangential stress rate relaxation*", abbreviated as "*tangential relaxation*". Here, note that the relaxation has to be diviatoric (cf. Rudnicki and Rice's [2]) and has to be directed towards in-between tangent and outward-normal to the subloading surface (cf. e.g. Kuroda and Tvergaard [5]). Then, let Eq. (12) be extended as

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{E}\mathbf{D} - \mathbf{E}\frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^p}\mathbf{M} - \overset{\circ}{\boldsymbol{\sigma}}^r, \tag{13}$$

while $\overset{\circ}{\boldsymbol{\sigma}}^r$ is called the *tangential relaxation stress rate* and let it be given as

$$\overset{\circ}{\boldsymbol{\sigma}}^r = S_r\left(\frac{\overset{\circ}{\boldsymbol{\sigma}}^*_t}{\|\overset{\circ}{\boldsymbol{\sigma}}^*_t\|} + d_n\mathbf{n}^*\right)\|\overset{\circ}{\boldsymbol{\sigma}}^*_t\|, \tag{14}$$

where the deviatoric-tangential stress rate $\overset{\circ}{\boldsymbol{\sigma}}^*_t$ is given as follows:

$$\overset{\circ}{\boldsymbol{\sigma}}^*_t = \overset{\circ}{\boldsymbol{\sigma}}^* - \mathrm{tr}(\mathbf{n}^*\overset{\circ}{\boldsymbol{\sigma}})\mathbf{n}^* = \overset{\circ}{\boldsymbol{\sigma}}^* - (\mathbf{n}^*\otimes\mathbf{n}^*)\overset{\circ}{\boldsymbol{\sigma}} = (\overline{\mathbf{I}}^* - \mathbf{n}^*\otimes\mathbf{n}^*)\overset{\circ}{\boldsymbol{\sigma}}, \tag{15}$$

$$\mathbf{n}^* \equiv \left(\frac{\partial f(\hat{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\right)^* \Big/ \left\|\left(\frac{\partial f(\hat{\boldsymbol{\sigma}},\mathbf{H})}{\partial\boldsymbol{\sigma}}\right)^*\right\| = \frac{\mathbf{N}^*}{\|\mathbf{N}^*\|} \quad (\|\mathbf{n}^*\| = 1)\,. \tag{16}$$

$(\ )^*$ stands for the deviatoric component and $\overline{\mathbf{I}}^*$ is the fourth-order *deviatoric transformation tensor*, i.e,

$$\overline{I}^*_{ijkl} \equiv \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}\,. \tag{17}$$

The material function S_r is a monotonically decreasing function of R, simply given by

$$S_r = \xi R^n, \tag{18}$$

where n is a material constant and ξ is a material parameter which is a function of stress and plastic internal variables in general: a material constant for metals and a function of stress for frictional materials. d_n is a material constant by which the relaxation has the direction in-between the tangent and the inward-normal to the subloading surface.

The strain rate is expressed in terms of the stress rate from Eqs. (8), (13) and (14) as

$$\mathbf{D} = \mathbf{E}^{-1}\overset{\circ}{\boldsymbol{\sigma}} + \frac{\mathrm{tr}(\mathbf{N}\overset{\circ}{\boldsymbol{\sigma}})}{M^p}\mathbf{M} + S_r\mathbf{E}^{-1}(\overset{\circ}{\boldsymbol{\sigma}}^*_t + d_n\|\overset{\circ}{\boldsymbol{\sigma}}^*_t\|\mathbf{n}^*). \tag{19}$$

Then, the strain rate is additively decomposed into the elastic strain rate $\mathbf{D}^e$, the plastic strain rate $\mathbf{D}^p$ and the tangential strain rate $\mathbf{D}^t$, i.e.

$$\mathbf{D} = \mathbf{D}^e + \mathbf{D}^p + \mathbf{D}^t , \tag{20}$$

while the tangential strain rate is given for Eq. (19) as follows:

$$\mathbf{D}^t = S_r \mathbf{E}^{-1}(\overset{\circ}{\boldsymbol{\sigma}}{}^*_t + d_n \mathrm{P}\overset{\circ}{\boldsymbol{\sigma}}{}^*_t \mathrm{P}\mathbf{n}^*) . \tag{21}$$

The positive proportionality factor λ in the flow rule (7) is expressed in terms of strain rate, rewriting λ by Λ, from Eq. (19) as follows:

$$\Lambda = \frac{\mathrm{tr}(\mathbf{NED}) - S_r d_n \mathrm{tr}(\mathbf{Nn}^*)\left\|\overset{\circ}{\boldsymbol{\sigma}}{}^*_t\right\|}{M^p + \mathrm{tr}(\mathbf{NEM})} . \tag{22}$$

The loading criterion for the plastic strain rate is given as follows [6]:

$$\left.\begin{aligned} \mathbf{D}^p \neq \mathbf{0}: \ \Lambda > 0, \\ \mathbf{D}^p = \mathbf{0}: \ \Lambda \leq 0, \end{aligned}\right\} \tag{23}$$

while the tangential strain rate $\mathbf{D}^t$ is always induced for $\overset{\circ}{\boldsymbol{\sigma}}{}^*_t \neq \mathbf{0}$. Eq. (19) is rate-nonlinear and thus an inverse expression becomes rather complicated form.

Now, consider the simple case d_n=0, i.e.

$$\overset{\circ}{\boldsymbol{\sigma}}{}^r = S_r \overset{\circ}{\boldsymbol{\sigma}}{}^*_t , \tag{24}$$

for which it hold that

$$\Lambda = \frac{\mathrm{tr}(\mathbf{NED})}{M^p + \mathrm{tr}(\mathbf{NEM})} , \tag{25}$$

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{ED} - \Lambda \mathbf{EM} - S_r \overset{\circ}{\boldsymbol{\sigma}}{}^*_t . \tag{26}$$

It is obtained from Eq. (26) that

$$\overset{\circ}{\boldsymbol{\sigma}}{}^* = \bar{\mathbf{I}}^* \mathbf{ED} - \Lambda \bar{\mathbf{I}}^* \mathbf{EM} - S_r \overset{\circ}{\boldsymbol{\sigma}}{}^*_t , \quad \mathrm{tr}(\mathbf{n}^* \overset{\circ}{\boldsymbol{\sigma}}{}^*)\mathbf{n}^* = \mathrm{tr}(\mathbf{n}^* \mathbf{ED})\mathbf{n}^* - \Lambda \mathrm{tr}(\mathbf{n}^* \mathbf{EM})\mathbf{n}^* . \tag{27}$$

The subtraction of the second equation from the first equation in Eq. (27) leads to

$$\overset{\circ}{\boldsymbol{\sigma}}{}^*_t = \frac{1}{1+S_r}[\bar{\mathbf{I}}^* \mathbf{ED} - \Lambda\{\bar{\mathbf{I}}^* \mathbf{EM} - \mathrm{tr}(\mathbf{n}^* \mathbf{EM})\mathbf{n}^*\} - \mathrm{tr}(\mathbf{n}^* \mathbf{ED})\mathbf{n}^*] . \tag{28}$$

The substitution of Eq. (28) into Eq. (26) leads to the inverse expression:

$$\overset{\circ}{\boldsymbol{\sigma}} = \mathbf{ED} - \Lambda \mathbf{EM} - \frac{S_r}{1+S_r}[\bar{\mathbf{I}}^* \mathbf{ED} - \Lambda\{\bar{\mathbf{I}}^* \mathbf{EM} - \mathrm{tr}(\mathbf{n}^* \mathbf{EM})\mathbf{n}^*\} - \mathrm{tr}(\mathbf{n}^* \mathbf{ED})\mathbf{n}^*] . \tag{29}$$

As shown in the above formulations, the analytical expression of stress rate in terms of strain rate is derived for Eq. (13) with Eq. (24) and the general elastic modulus tensor **E**. On the other hand, the expression of stress rate in terms of strain rate is obtained only for the Hooke's law in the models of Rudnicki and Rice [2] and Hashiguchi and Tsutsumi [1] ($\mathbf{D}^t = C_r \overset{\circ}{\boldsymbol{\sigma}}{}_t^*$; C_r : material parameter).

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Appendix – Relations of Corotational Rates

We should not say easily which spin is best for the corotational rate. For instance, compare the *Jaumann rate* and the *corotational rate with the plastic spin*. The elastoplastic constitutive equation adopting the latter is described as follows:

$$\overset{\bullet}{\boldsymbol{\sigma}} - (\mathbf{W} - \mathbf{W}^p)\boldsymbol{\sigma} + \boldsymbol{\sigma}(\mathbf{W} - \mathbf{W}^p) = \mathbf{E}\mathbf{D} - \mathbf{E}\lambda\mathbf{M} = \mathbf{C}^{ep}\mathbf{D} \tag{a.1}$$

where $\mathbf{C}^{ep}$ is the elastoplastic stiffness modulus tensor given by

$$\mathbf{C}^{ep} \equiv \mathbf{E} - \frac{1}{M^p}\mathbf{E}\mathbf{M} \otimes \mathbf{N}\mathbf{E}\,. \tag{a.2}$$

$\mathbf{W}$ is the continuum spin (skew-symmetric part of the velocity gradient) and $\mathbf{W}^p$ is the plastic spin the concrete form of which may be given as follows:

$$\mathbf{W}^p = \mu(\boldsymbol{\sigma}\mathbf{D}^p - \mathbf{D}^p\boldsymbol{\sigma})\,, \tag{a.3}$$

where μ is the material parameter.

Eq. (a.1) can be expressed as

$$\overset{\bullet}{\boldsymbol{\sigma}} - \mathbf{W}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{W} = \mathbf{E}\mathbf{D} - \mathbf{D}^p - \mu(\boldsymbol{\sigma}\mathbf{D}^p - \mathbf{D}^p\boldsymbol{\sigma})\boldsymbol{\sigma} + \mu\boldsymbol{\sigma}(\boldsymbol{\sigma}\mathbf{D}^p - \mathbf{D}^p\boldsymbol{\sigma})$$
$$= \mathbf{E}\mathbf{D} - \lambda\{\mathbf{M} + \mu(2\boldsymbol{\sigma}\mathbf{M}\boldsymbol{\sigma} - \mathbf{M}\boldsymbol{\sigma}^2 - \boldsymbol{\sigma}^2\mathbf{M})\}\,. \tag{a.4}$$

Adopting the Jaumann rate

$$\overset{\circ}{\boldsymbol{\sigma}}{}^J \equiv \overset{\bullet}{\boldsymbol{\sigma}} - \mathbf{W}\boldsymbol{\sigma} + \boldsymbol{\sigma}\mathbf{W}, \tag{a.5}$$

and setting

$$\bar{\mathbf{M}} \equiv \mathbf{M} + \mu(2\boldsymbol{\sigma}\mathbf{M}\boldsymbol{\sigma} - \mathbf{M}\boldsymbol{\sigma}^2 - \boldsymbol{\sigma}^2\mathbf{M}), \tag{a.6}$$

Eq. (a.4) can be expressed as

$$\overset{\circ}{\boldsymbol{\sigma}}{}^J = \mathbf{E}\mathbf{D} - \lambda\mathbf{E}\bar{\mathbf{M}} = \mathbf{E}\mathbf{D} - \mathbf{E}\frac{\mathrm{tr}(\mathbf{N}\mathbf{E}\mathbf{D})}{M^p}\bar{\mathbf{M}} = \bar{\mathbf{C}}^{ep}\mathbf{D}, \tag{a.7}$$

with the plastic strain rate

$$\mathbf{D}^p = \lambda\bar{\mathbf{M}}, \tag{a.8}$$

where

$$\bar{\mathbf{C}}^{ep} \equiv \mathbf{E} - \frac{1}{M^p}\mathbf{E}\bar{\mathbf{M}} \otimes \mathbf{N}\mathbf{E}\,. \tag{a.9}$$

Besides, Khan and Huang [7] showed that the spin with the plastic spin and the *Green-Naghdi spin* $\boldsymbol{\Omega}^R$ just coincide with each other for the rigid plastic materials, i.e.

$$\mathbf{W} - \mathbf{W}^p = \boldsymbol{\Omega}^R, \tag{a.10}$$

where

$$\boldsymbol{\Omega}^R \equiv \dot{\mathbf{R}}\,\mathbf{R}^{\mathrm{T}}. \tag{a.11}$$

$\mathbf{R}$ is the rotational component obtained from the polar decomposition of the deformation gradient. Then, it holds that

$$\overset{\bullet}{\boldsymbol{\sigma}} - \boldsymbol{\Omega}^R\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{\Omega}^R = \mathbf{C}^{ep}\mathbf{D}\,. \tag{a.12}$$

It should be noted that Eq. (a.1) adopting the plastic spin $\mathbf{W} - \mathbf{W}^p$, Eq. (a.5) (Jaumann rate) adopting the continuum spin $\mathbf{W}$ and Eq. (a.12) adopting the Green-Naghdi spin $\boldsymbol{\Omega}^R$ describe the same material behavior in the similar or same form with different or same elastoplastic stiffness modulus tensor. Eventually, it would be nonsense to say which spin is best but the selection of spin has to be determined based on the concrete formulation of elastoplastic stiffness tensor.