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The Sequence-of-Bifurcation Approach Towards Understanding Turbulent Flows

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Summary

Non-equilibrium fluid systems that are homogeneous in two spatial dimensions and in time are considered. They offer a maximum of symmetries the breakings of which are needed to identify bifurcations. The best known examples are the Taylor-Couette system and Rayleigh-Bénard convection. While the first instability generically occurs in the form of rolls or stripes, the physical properties of the system are reflected by secondary bifurcations. The latter usually do not exhaust the available symmetries and tertiary and quaternary bifurcations can be investigated and compared with experimental observations as will be demonstrated in the case of the Rayleigh-Bénard problem. The structures introduced by higher bifurcations often persist as coherent structures in the turbulent state of the respective system.

Introduction

The most common approach towards understanding turbulent fluid flow is based on statistical analysis. Since the details of the nearly random velocity fields observed in laboratory experiments or in numerical simulations are of little interest, one usually tries to characterize turbulent systems by their statistical properties and by their time averaged properties in particular. Since it is generally accepted that the basic Navier-Stokes equations (NSE) of motion provide the correct basis for the description of turbulent fluid flow it is regrettable that rather little information from the basic dynamical balances enters into the statistical analysis of turbulence. The goal of this article is to demonstrate that relatively simple spatially periodic solutions of the NSE can be quite useful for the understanding of typical dynamical mechanisms operating in turbulent fluid systems. These "regular" solutions, - as we shall call them in distinction to the turbulent solutions -, may well be unstable and thus not observable in experiments. But there are many situations where regular solutions are stable with respect to infinitesimal disturbances and where their basins of attraction in the solution space are just too small to permit their realization in an experiment.

In order to study the regular solutions it is convenient to restrict the attention to systems that are homogeneous with respect two spatial dimensions. We shall also use homogeneity in time by assuming constant external conditions. Since we are considering non-equilibrium systems there must always be a dimension in which an energy flux enters and exits the system. In figure 1 some typical systems are

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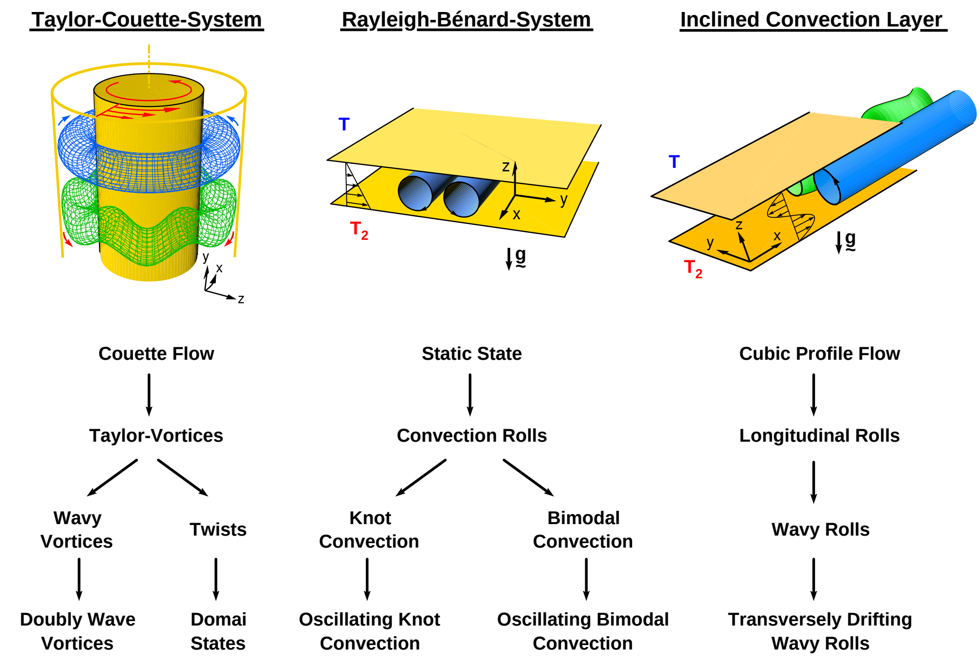

Figure 1: Examples for sequences of bifurcations in fluid dynamical systems.

displayed. While the basic state or primary solution reflects the homogeneity of the configuration of the problem, spontaneous symmetry breakings characterize the solutions bifurcating from the basic state as the control parameter increases. Unless the bifurcation is subcritical these bifurcating solutions can usually be realized in experiments when homogeneity is approached to a sufficient degree. The bifurcating solution can be steady or oscillatory, but generically it is two-dimensional, i.e. it assumes the form of rolls or stripes. With further increasing control parameter secondary bifurcations are likely to occur which will break additional symmetries by introducing, for instance, a second spontaneous wavenumber along the axis of the rolls. While the secondary solution introduced by the first bifurcation is rather similar in all cases owing to its two-dimensional nature, the tertiary, quaternary and higher order states introduced by the second, third and higher bifurcations are specific to the respective physical system and may also hange in dependence on parameters of the system. For this reason different branches of bifurcation sequences have been indicated in figure 1 in the cases of the Taylor-Couette-system and the Rayleigh-Bénard layer. Further branches exist which have not been mentioned. The approach to be described in the following has been developed to follow these bifurcation branches and to identify the physical mechanisms that are the cause of the bifurcations.

Mathematical Description of Secondary Solutions and their Instabilities

In keeping with the general character of the sequence-of-bifurcations approach we shall not consider specific equations here. Instead we consider the more abstract mathematical problem of the form

$$\underline{\boldsymbol{L}}\boldsymbol{X} + R\underline{\boldsymbol{K}}\boldsymbol{X} - \underline{\boldsymbol{M}}\frac{\partial}{\partial t}\boldsymbol{X} = \underline{\boldsymbol{N}}(\boldsymbol{X},\boldsymbol{X}) \tag{1}$$

where the vector valued variable $\boldsymbol{X}$ describes the deviation from the basic state which exhibits the same degree of homogeneity as the external conditions of the system. For this reason equation (1) is homogeneous and homogeneous boundary conditions must be obeyed by $\boldsymbol{X}$. $\underline{\boldsymbol{L}}, \underline{\boldsymbol{K}}, \underline{\boldsymbol{M}}$ are linear operators involving partial derivatives, while $\underline{\boldsymbol{N}}(\boldsymbol{X},\boldsymbol{X})$ is the nonlinear part which typically is quadratic in the variable $\boldsymbol{X}$. R denotes a control parameter such as the Rayleigh or Reynolds number. Although the surfaces of homogeneity could be spherical or cylindrical, we shall restrict the attention for simplicity to planar surfaces. Accordingly we introduce a Cartesian system of coordinates x, y, z with the z-coordinate in the inhomogeneous dimension. The operators of equation (1) may thus depend on z, but not on x, y or time t. In order to investigate the stability of the basic state with respect to infinitesimal disturbances we neglect the right hand side of equation (1) and look for solutions of the general form

$$\boldsymbol{X} = \exp\{i\boldsymbol{l}\cdot\boldsymbol{r} + \sigma t\}\boldsymbol{G}(\boldsymbol{l}, z) \tag{2}$$

where $\boldsymbol{l}$ is an arbitrary wavevector parallel to the surfaces of homogeneity and σ is the growth rate which represents the complex eigenvalue of the linear homogeneous equation (1) with vanishing right hand side. Of physical interest is the lowest value of R for which an eigenvalue σ exists as a function of $\boldsymbol{l}$ with vanishing real part σ_r. The corresponding values R_c of R and $\boldsymbol{l}_c$ of $\boldsymbol{l}$ are called critical control parameter and critical wave vector, respectively. For R exceeding R_c disturbances of the form (2) will grow, but their amplitudes will saturate owing to the action of the nonlinearity on the right hand side of equation (1). In the case of a supercritical bifurcation, - which we shall assume -, the saturated amplitude varies smoothly with $R - R_c$. For the description of this solution we orientate the coordinate system such that y points in the direction of $\boldsymbol{l}_c$ and assume $\sigma_i = 0$. The treatment of the case $\sigma_i \neq 0$ is analogous though slightly more complex. We thus arrive at the representation

$$X_i = \sum_{m,n} a_{mn}^{(i)} \exp\{im\alpha y\} G_n^{(i)}(z) \tag{3}$$

for the steady bifurcating solution in the form of rolls or stripes. The vector functions $G_n^{(i)}$ denote a complete system satisfying all boundary conditions and the wavenumber α typically is set equal to $\mid \boldsymbol{l}_c \mid$. But when R exceeds R_c there usually exists a neighborhood of wavevectors $\boldsymbol{l}$ around $\boldsymbol{l}_c$ for which solutions of the form (3) can be obtained. Although solutions of the form (3) are likely to exist for all values $R > R_c$, they will in general become unstable as R increases much beyond

Table 1: Symmetry properties of two-dimensional rolls (w refers to the velocity component in the z-direction)

A	translation in time:	$\partial w/\partial t = 0$
B	translation along roll axis:	$\partial w/\partial x = 0$
C	transverse periodicity:	$w(y + 2\pi/\alpha_1 z) = w(y, z)$
D	transverse reflection:	$w(-y, z) = w(y, z)$ or $a_{-mn} = a_{mn}$
E	inversion about roll axis:	$w(y + \frac{\pi}{\alpha}, z) = -w(y, -z)$ or $a_{mn} = 0$ for odd $m + n$

R_c. In order to investigate the stability with respect to infinitesimal disturbances $\tilde{\boldsymbol{X}}$ we must solve the linear homogeneous problem

$$\underline{\boldsymbol{L}}\tilde{\boldsymbol{X}} + R\underline{\boldsymbol{M}}\tilde{\boldsymbol{X}} + \underline{\boldsymbol{V}}\frac{\partial}{\partial t}\tilde{\boldsymbol{X}} = \boldsymbol{N}(\tilde{\boldsymbol{X}}, \boldsymbol{X}) + \boldsymbol{N}(\boldsymbol{X}, \tilde{\boldsymbol{X}}) \tag{4}$$

Since $\boldsymbol{X}$ is steady and periodic in y a Floquet ansatz

$$\tilde{X}_i = \exp\{ibx + idy + \sigma t\} \sum_{m,n} \tilde{a}^{(i)}_{nm} \exp\{im\alpha y\} G^{(i)}_n(z) \tag{5}$$

can be assumed without loosing generality. For a given solution of the form (3) the eigenvalues σ must be determined in dependence on the wavenumbers b and d. Whenever there exists a σ with positive real part σ_r the steady roll solution $\boldsymbol{X}$ is unstable. If all σ_r are negative or zero the solution $\boldsymbol{X}$ is regarded as stable. There exist always the neutral disturbance $\tilde{\boldsymbol{X}} = \partial \boldsymbol{X}/\partial y$ as solution of equation (4) with $\sigma = 0$ which corresponds to an infinitesimal translation of the steady roll solution perpendicular to its axis. In order to classify the growing disturbances of the form (5) it is convenient to consider all symmetries of the solution (3) which could possibly be broken by the instability. Among the symmetries of rolls listed in table 1 the first three are common to all solutions of the form (3). The additional symmetry D is found for convection rolls in a Rayleigh-Bénard layer or for Taylor vortices between differentially rotating cylinders. In the case of symmetry with respect to the midplane of the layer symmetry E holds as well. In table 2 typical instabilities of rolls are listed that have been found in the Rayleigh-Bénard case and in other systems. For details we refer to [1],[2],[3],[4]. Some of the instabilities listed in table 2 do not lead to new states of convection, but generate convection rolls with different wavelengths corresponding to values of α in the stable domain, as, for example, in the case of the Eckhaus instability or in the case of the cross roll instability at lower values of the Rayleigh number. Of special interest, however, are instabilities evolving into three-dimensional structures which will be considered in the next section.

Table 2: Symmetries Broken by Bifurcations from Rolls

Broken Symmetries		A	B	C	D	E	
Properties of disturbances		$\sigma_i \neq 0$	$b \neq 0$	$d \neq 0$	$\tilde{a}_{mn} \neq \tilde{a}_{-mn}$	$\tilde{a}_{mn} \neq 0$ for $m+n =$ odd	Remarks
Eckhaus Instab.				X	X		
Crossroll Instab.	CR		X			X	} differ by value of b
Knot-Instability	KN		X			X	
Even Blob-Instab.	EB	X	X				
Odd Blob-Instab.	OB	X	X			X	
Oscillatory Instab.	OS	X	X		X		
Zig-Zag-Instability	ZZ		X		X	{	occurs also as wavy instability of Taylor vortices
Skewed Varic. Inst.	SV		X	X	X		occurs as Küppers-Lortz instability in a rotating convection layer
Osc. Skewed Var. Inst.		X	X	X	X		

Three-Dimensional Solutions Emerging from Secondary and Higher-Order Bifurcations

The mathematical analysis of tertiary solutions follows in close analogy to the analysis of rolls. After the Galerkin representation

$$X_i = \sum_{l,m,n} a_{lmn}^{(i)} \exp\{il\alpha_x x + im\alpha_y y\} G_n^{(i)}(z) \tag{6}$$

has been introduced where $\alpha_x = b, \alpha_y = \alpha$ have been used, nonlinear algebraic equations for the coefficients $a_{lmn}^{(i)}$ can be obtained by the projection of the basic equations onto the space of the expansion functions used in the representation (6). The algebraic equations can then be solved by a Newton-Raphson method after a suitable truncation of the summations in expression (6) has been employed. Representation (6) is applicable for solutions evolving from instabilities of rolls with $d = 0$ and vanishing value σ_i. But for $\sigma_i \neq 0$ representation (6) can also be used since oscillatory instabilities of rolls typically evolve into traveling waves. In that case x must be replaced by $\hat{x} = x - ct$ and the method of solution proceeds just as in the case of steady tertiary solution. Even in the case of a finite values of d of the strongest growing instability the representation (6) can still be used after $\alpha = \alpha_y$ has been replaced by a fractional value, $\hat{\alpha}_y = \alpha/p$ where the integers p and q is chosen such that $\alpha q/p$ or $\alpha(1-q/p)$ approximates d. The evolution of subharmonic instabilities with $d = \alpha/2$ can most easily be analyzed in this way [5].

Tertiary solutions still exhibit a number of symmetries and their instabilities can be investigated with a Floquet ansatz analogous to (5). The sequence-of bifurcations approach can be continued to quaternary and higher order solutions until all available symmetries are exhausted. Theoretical solutions in the case of convection [6] compare well with experimental observations.

Reference

1. Busse, F.H. (1967): "On the stability of two-dimensional convection in a layer heated from below", *J. Math. Phys.*, Vol. 46, pp.140-150.

2. Clever, R.M., and Busse, F.H. (1974): "Transition to time-dependent convection", *J. Fluid Mech.* Vol. 65, pp.625-645.
3. Busse, F.H., and Clever, R.M.(1979): "Instabilities of convection rolls in a fluid of moderate Prandtl number", *J. Fluid Mech.* Vol. 91, pp.319-335.
4. Bolton, E.W., Busse, F.H., and Clever, R.M. (1986): "Oscillatory instabilities of convection rolls at intermediate Prandtl numbers", *J. Fluid Mech.* Vol. 164, pp.469-485.
5. Nagata, M., and Busse, F.H. (1983): "Three-dimensional tertiary motions in a plane shear layer", *J. Fluid Mech.* Vol.135, pp.1-26.
6. Clever, R.M., and Busse, F.H. (1994): "Steady and oscillatory bimodal convection", *J. Fluid Mech.* Vol. 271, pp.103-118.