Boundary Integral Equations for Thermoelastic Plates

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Summary

The usefulness of plate theories resides in that they reduce complicated three-dimensional problems to simpler ones in two dimensions without compromising the essential information needed in the study of the phenomenon of bending. In this paper we solve the initial value problem governing the motion of an infinite thermoelastic plate, finding the solution in terms of "initial" and "area" potentials. This is a fundamental preliminary step in the construction of boundary element methods for finite plates.

Formulation of the Problem

Consider an infinite elastic plate of thickness $h_0 = \text{const} > 0$, which occupies a region $\mathbb{R}^2 \times [-h_0/2, h_0/2]$ in $\mathbb{R}^3$. The displacement vector at a generic point x' at $t \geq 0$ is $v(x', t) = (v_1(x', t), v_2(x', t), v_3(x', t))^{\mathrm{T}}$, where the superscript T signifies matrix transposition. Let $x' = (x, x_3)$, with $x = (x_1, x_2) \in \mathbb{R}^2$. In plate models with transverse shear deformation it is assumed [1] that $v(x', t) = (x_3 u_1(x, t), x_3 u_2(x, t), u_3(x, t))^{\mathrm{T}}$. If thermal effects are taken into account, we also introduce the "averaged" temperature across thickness [2], denoted by u_4. Then the function $U(x, t)$, $U = (u^{\mathrm{T}}, u_4)^{\mathrm{T}}$, $u = (u_1, u_2, u_3)^{\mathrm{T}}$, satisfies the equation

$$\mathcal{B}_0 \partial_t^2 U(x,t) + \mathcal{B}_1 \partial_t U(x,t) + \mathcal{A} U(x,t) = \mathcal{Q}(x,t), \quad (x,t) \in G; \qquad (1)$$

here $G = \mathbb{R}^2 \times (0, \infty)$, $\mathcal{B}_0 = \text{diag}\{\rho h^2, \rho h^2, \rho, 0\}$, $\partial_t = \partial/\partial t$, $\rho > 0$ is the constant density of the material,

$$\mathcal{B}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \eta\partial_1 & \eta\partial_2 & 0 & \varkappa^{-1} \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} & A & & h^2\gamma\partial_1 \\ & & & h^2\gamma\partial_2 \\ & & & 0 \\ 0 & 0 & 0 & -\Delta \end{pmatrix},$$

$$A = \begin{pmatrix} -h^2\mu\Delta - h^2(\lambda+\mu)\partial_1^2 + \mu & -h^2(\lambda+\mu)\partial_1\partial_2 & \mu\partial_1 \\ -h^2(\lambda+\mu)\partial_1\partial_2 & -h^2\mu\Delta - h^2(\lambda+\mu)\partial_2^2 + \mu & \mu\partial_2 \\ -\mu\partial_1 & -\mu\partial_2 & -\mu\Delta \end{pmatrix},$$

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$\partial_\alpha = \partial/\partial_\alpha$, $\alpha = 1,2$, $\eta, \varkappa$, and γ are positive constants, λ and μ are the Lamé constants of the material satisfying $\lambda+\mu > 0$, $\mu > 0$, and $\mathcal{Q}(x,t) = (q(x,t)^{\mathrm{T}}, q_4(x,t))^{\mathrm{T}}$, where $q(x,t) = (q_1(x,t), q_2(x,t), q_3(x,t))^{\mathrm{T}}$ is a combination of the forces and moments acting on the plate and its faces and $q_4(x,t)$ is a combination of the averaged heat source density and the temperature and heat flux on the faces.

The classical initial value (Cauchy) problem for (1) consists in finding a function $U \in \mathrm{C}^2(G)$, $u \in \mathrm{C}^1(\bar{G})$, $u_4 \in \mathrm{C}(\bar{G})$, satisfying (1) and

$$U(x,0) = U_0(x), \quad \partial_t u(x,0) = \psi(x), \quad x \in \mathbb{R}^2, \tag{2}$$

where $U_0 = (\varphi^{\mathrm{T}}, \theta)^{\mathrm{T}}$, $\varphi = (\varphi_1, \varphi_2, \varphi_3)^{\mathrm{T}}$, and $\psi = (\psi_1, \psi_2, \psi_3)^{\mathrm{T}}$ are given.

$\mathbb{H}_{1,\kappa}(G)$, $\kappa > 0$, is the space of four-component distributions $U(x,t)$ on G with norm $\|U\|^2_{1,\kappa;G} = \int\limits_G e^{-2\kappa t}\big\{|U(x,t)|^2 + |\partial_t U(x,t)|^2 + \sum\limits_{i=1}^{4} |\nabla u_i(x,t)|^2\big\}\, dx\, dt$. An equivalent norm is $\big\{\int\limits_G e^{-2\kappa t}\big[(1+|\xi|)^2 |\tilde{U}(\xi,t)|^2 + |\partial_t \tilde{U}(\xi,t)|^2\big]\, d\xi\, dt\big\}^{1/2}$, where $\tilde{U}(\xi,t) = (\tilde{u}(\xi,t)^{\mathrm{T}}, \tilde{u}_4(\xi,t))^{\mathrm{T}}$, $\tilde{u}(\xi,t) = (\tilde{u}_1(\xi,t), \tilde{u}_2(\xi,t), \tilde{u}_3(\xi,t))^{\mathrm{T}}$, is the Fourier transform of $U(x,t)$ with respect to x. Below we do not distinguish between equivalent norms and denote them by the same symbol.

The variational formulation of problem (1), (2) consists in finding $U \in \mathbb{H}_{1,\kappa}(G)$ for some $\kappa > 0$, which satisfies

$$\begin{aligned}
&\int\limits_0^\infty \big[a(u,w) - (B_0^{1/2}\partial_t u, B_0^{1/2}\partial_t w)_0 + h^2\gamma\eta^{-1}\varkappa^{-1}(w_4, \partial_t u_4)_0 \\
&\quad + h^2\gamma\eta^{-1}(\nabla w_4, \nabla u_4)_0 - h^2\gamma(\nabla w_4, \partial_t u)_0 + h^2\gamma(\nabla u_4, w)_0\big]\, dt \\
&= (B_0\psi, \gamma_0 w)_0 + \int\limits_0^\infty \big[(q,w)_0 + h^2\gamma\eta^{-1}(w_4, q_4)_0\big]\, dt
\end{aligned}$$

for any $W \in \mathrm{C}_0^\infty(\bar{G})$, and $\gamma_0 U = U_0$, where $B_0 = \operatorname{diag}\{\rho h^2, \rho h^2, \rho\}$, $(\cdot\,,\cdot)$ is the inner product in $\mathbb{C}^m$, $(\cdot\,,\cdot)_0$ is the inner product in $\big[L^2(\mathbb{R}^2)\big]^m$ for any $m \in \mathbb{N}$, γ_0 is the continuous trace operator from the space of index $m \in \mathbb{N}$ with weight $\exp(-2\kappa t)$, $t > 0$, of functions in G, to the corresponding Sobolev space of index $m - 1/2$ of functions in $\mathbb{R}^2$, and $a(u,w) = 2\int\limits_{\mathbb{R}^2} E(u,w)\, dx$ is a sesquilinear form in which $E(u,u)$ is the potential energy density of the plate [1]. We remark that if $f \in \mathrm{C}^2(\mathbb{R}^2)$ and $g \in \mathrm{C}_0^\infty(\mathbb{R}^2)$, then $(Af, g)_0 = a(f,g)$.

Theorem 1. *Problem* (1), (2) *has at most one solution of class* $\mathbb{H}_{1,\kappa}(G)$.

The solution of (1), (2) is the sum of the solution of the problem for the homogeneous system (1) with the given initial data and of that for the nonhomogeneous system (1) with zero initial data.

The Homogenous Equation

First, let $\mathcal{Q}(x,t)=0$. Then we seek $U\in\mathbb{H}_{1,\kappa}(G)$ that satisfies

$$\int_0^\infty \big[a(u,w)-(B_0^{1/2}\partial_t u, B_0^{1/2}\partial_t w)_0+h^2\gamma\eta^{-1}\varkappa^{-1}(w_4,\partial_t u_4)_0 \\ +h^2\gamma\eta^{-1}(\nabla w_4,\nabla u_4)_0-h^2\gamma(\nabla w_4,\partial_t u)_0+h^2\gamma(\nabla u_4,w)_0\big]\,dt \\ =(B_0\psi,\gamma_0 w)_0\quad \forall W=(w^{\mathrm{T}},w_4)^{\mathrm{T}}\in \mathrm{C}_0^\infty(\bar G) \tag{3}$$

and $\gamma_0 U=U_0=(\varphi^{\mathrm{T}},\theta)^{\mathrm{T}}$.

We denote by $D(x,t)$ a matrix of fundamental solutions for (1) and define the "initial" potentials of the first kind of density $F(x)$, $F=(f^{\mathrm{T}},f_4)^{\mathrm{T}}$, $f=(f_1,f_2,f_3)^{\mathrm{T}}$,

$$\mathcal{J}(x,t)=(\mathcal{J}F)(x,t)=\int_{\mathbb{R}^2} D(x-y,t)F(y)dy,\quad (x,t)\in G,$$

and of the second kind of density $G(x)$, $G=(g^{\mathrm{T}},g_4)^{\mathrm{T}}$, $g=(g_1,g_2,g_3)^{\mathrm{T}}$,

$$\mathcal{E}(x,t)=(\mathcal{E}G)(x,t)=\int_{\mathbb{R}^2}\partial_t D(x-y,t)G(y)dy=\partial_t(\mathcal{J}G)(x,t),\ (x,t)\in G.$$

We write $\mathcal{J}=(j^{\mathrm{T}},j_4)^{\mathrm{T}}$, $j=(j_1,j_2,j_3)^{\mathrm{T}}$, and $\mathcal{E}=(e^{\mathrm{T}},e_4)^{\mathrm{T}}$, $e=(e_1,e_2,e_3)^{\mathrm{T}}$.

Lemma 2. (i) *If $f\in H_1(\mathbb{R}^2)$ and $f_4\in H_2(\mathbb{R}^2)$, then $\mathcal{J}F\in\mathbb{H}_{1,\kappa}(G)$ for any $\kappa>0$.*

(ii) *If $g\in H_3(\mathbb{R}^2)$ and $g_4\in H_4(\mathbb{R}^2)$, then $\mathcal{E}G\in\mathbb{H}_{1,\kappa}(G)$ for any $\kappa>0$.*

Lemma 3. (i) *If $f\in H_1(\mathbb{R}^2)$ and $f_4\in H_2(\mathbb{R}^2)$, then $\mathcal{J}F\in\mathbb{H}_{1,\kappa}(G)$ for any $\kappa>0$, $j(x,t)\to 0$, as $t\to 0$, in $H_2(\mathbb{R}^2)$, $j_4(x,t)\to\varkappa f_4(x)$, as $t\to 0$, in $H_1(\mathbb{R}^2)$, and $\mathcal{J}(x,t)$ satisfies*

$$\int_0^\infty \big[a(j,w)-(B_0^{1/2}\partial_t j, B_0^{1/2}\partial_t w)_0+h^2\gamma\eta^{-1}\varkappa^{-1}(w_4,\partial_t j_4)_0 \\ +h^2\gamma\eta^{-1}(\nabla w_4,\nabla j_4)_0-h^2\gamma(\nabla w_4,\partial_t j)_0+h^2\gamma(\nabla j_4,w)_0\big]\,dt \\ =(f,\gamma_0 w)_0\quad \forall\, W\in \mathrm{C}_0^\infty(\bar G).$$

(ii) *If* $g \in H_3(\mathbb{R}^2)$ *and* $g_4 \in H_4(\mathbb{R}^2)$, *then* $\mathcal{E}G \in \mathbb{H}_{1,\kappa}(G)$ *for any* $\kappa > 0$, $e(x,t) \to B_0^{-1}g(x)$, *as* $t \to 0$, *in* $H_3(\mathbb{R}^2)$, $e_4(x,t) \to -(\rho h^2)^{-1}\varkappa\eta \operatorname{div} g(x) + \varkappa^2\Delta g_4(x)$, *as* $t \to 0$, *in* $H_2(\mathbb{R}^2)$, *and* $\mathcal{E}(x,t)$ *satisfies*

$$\int_0^\infty \big[a(e,w) - (B_0^{1/2}\partial_t e, B_0^{1/2}\partial_t w)_0 + h^2\gamma\eta^{-1}\varkappa^{-1}(w_4,\partial_t e_4)_0 + h^2\gamma\eta^{-1}(\nabla w_4,\nabla e_4)_0 - h^2\gamma(\nabla w_4,\partial_t e)_0 + h^2\gamma(\nabla e_4, w)_0\big]\,dt = \varkappa h^2\gamma(g_4, \operatorname{div}(\gamma_0 w))_0 \quad \forall W \in \mathrm{C}_0^\infty(\bar{G}).$$

Theorem 2. *If* $\varphi \in H_3(\mathbb{R}^2)$, $\theta \in H_2(\mathbb{R}^2)$, $\psi \in H_1(\mathbb{R}^2)$, *and* $f = B_0\psi$, $f_4 = \varkappa^{-1}\theta + \eta \operatorname{div}\varphi$, $g = B_0\psi$, *and* $g_4 = 0$, *then* $\mathcal{J}F + \mathcal{E}G$ *is the solution of* (3) *in* $\mathbb{H}_{1,\kappa}(G)$ *for any* $\kappa > 0$,

$$\gamma_0(\mathcal{J}F + \mathcal{E}G) = (\varphi^{\mathrm{T}}, \theta)^{\mathrm{T}},$$

and

$$\|\mathcal{J}F + \mathcal{E}G\|_{1,\kappa;G} \le c\big\{\|\varphi\|_3 + \|\theta\|_2 + \|\psi\|_1\big\}.$$

Let $\mathbb{H}'_{1,\kappa}(G)$ be the space that coincides with $\mathbb{H}_{1,\kappa}(G)$ as a set but is equipped with the norm

$$\|U\|'_{1,\kappa;G} = \bigg\{\int_G e^{-2\kappa t}\big[(1+|\xi|)^2|\tilde{U}(\xi,t)|^2 + |\partial_t\tilde{u}(\xi,t)|^2\big]d\xi dt\bigg\}^{1/2}.$$

Theorem 3. *If*

$$\varphi \in H_{m+1}(\mathbb{R}^2), \quad \theta \in H_m(\mathbb{R}^2), \quad \psi \in H_m(\mathbb{R}^2), \quad m = 1,2,$$
$$\varphi \in H_{2m-1}(\mathbb{R}^2), \quad \theta \in H_{2m-2}(\mathbb{R}^2), \quad \psi \in H_{2m-3}(\mathbb{R}^2), \quad m \ge 3,$$

and $f = B_0\psi$, $f_4 = \varkappa^{-1}\theta + \eta \operatorname{div}\varphi$, $g = B_0\varphi$, *and* $g_4 = 0$, *then* $\mathcal{J}F + \mathcal{E}G$ *is the solution of* (3) *in* $\mathbb{H}'_{m,\kappa}(G)$ *for any* $\varkappa > 0$,

$$\gamma_0(\mathcal{J}F + \mathcal{E}G) = (\varphi^{\mathrm{T}}, \theta)^{\mathrm{T}},$$

and

$$\|\mathcal{J}F + \mathcal{E}G\|'_{m,\kappa;G} \le c(\|\varphi\|_{m+1} + \|\theta\|_m + \|\psi\|_m), \quad m = 1,2,$$
$$\|\mathcal{J}F + \mathcal{E}G\|'_{m,\kappa;G} \le c(\|\varphi\|_{2m-1} + \|\theta\|_{2m-2} + \|\psi\|_{2m-3}), \quad m \ge 3.$$

Homogeneous Boundary Conditions

Now let $\varphi(x) = \theta(x) = \psi(x) \equiv 0$. Then we seek $U \in \mathbb{H}_{1,\kappa}(G)$ that satisfies

$$\int_0^\infty \big[a(u,w) - (B_0^{1/2}\partial_t u, B_0^{1/2}\partial_t w)_0 + h^2\gamma\eta^{-1}\varkappa^{-1}(w_4,\partial_t u_4)_0 + h^2\gamma\eta^{-1}(\nabla w_4,\nabla u_4)_0 - h^2\gamma(\nabla w_4,\partial_t u)_0 + h^2\gamma(\nabla u_4, w)_0\big]\,dt$$
$$= \int_0^\infty \big[(q,w)_0 + h^2\gamma\eta^{-1}(w_4,q_4)_0\big]\,dt \quad \forall\, W \in \mathrm{C}_0^\infty(\bar{G}) \tag{4}$$

and $\gamma_0 U = 0$.

We introduce the so-called area potential $\mathcal{U}(x,t)$ of density $Q(x,t)$, $Q = (q^{\mathrm{T}}, q_4)^{\mathrm{T}}$, $q = (q_1,q_2,q_3)^{\mathrm{T}}$, of class $\mathrm{C}_0^\infty(G)$, by

$$\mathcal{U}(x,t) = (\mathcal{U}Q)(x,t) = \int_G D(x-y,t-\tau)Q(y,\tau)\,dy\,d\tau, \quad (x,t) \in G.$$

We recall that $H_m(\mathbb{R}^2)$ is the (vector or scalar) standard Sobolev space with index $m \in \mathbb{R}$ and norm

$$\|u\|_m = \Big\{ \int_{\mathbb{R}^2} (1+|\xi|^2)^m |\tilde{u}(\xi)|^2\,d\xi \Big\}^{1/2}.$$

Let $H_{m,p}(\mathbb{R}^2)$, $m \in \mathbb{R}$, $p \in \mathbb{C}$, be the space that coincides with $H_m(\mathbb{R}^2)$ as a set but is endowed with the norm

$$\|u\|_{m,p} = \Big\{ \int_{\mathbb{R}^2} (1+|\xi|^2+|p|^2)^m |\tilde{u}(\xi)|^2\,d\xi \Big\}^{1/2}.$$

We fix $\kappa > 0$ and consider the spaces $\mathcal{H}^{\mathcal{L}}_{m,k,\kappa}(\mathbb{R}^2)$ and $H^{\mathcal{L}}_{m,k,\kappa}(\mathbb{R}^2)$, $k \in \mathbb{R}$, of functions $\hat{u}(x,p)$ with the following properties:

(i) $\hat{u}(x,p)$, as a mapping from $\mathbb{C}_\kappa$ to $H_m(\mathbb{R}^2)$, is holomorphic;

(ii) $\hat{u} \in \mathcal{H}^{\mathcal{L}}_{m,k,\kappa}(\mathbb{R}^2)$ satisfies

$$[\hat{u}]^2_{m,k,\kappa} = \sup_{\sigma>\kappa} \int_{-\infty}^{\infty} (1+|p|^2)^k \|\breve{u}(\xi,p)\|^2_m\,d\tau < \infty; \tag{5}$$

$\hat{u} \in H^{\mathcal{L}}_{m,k,\kappa}(\mathbb{R}^2)$ satisfies

$$\|\hat{u}\|^2_{m,k,\kappa} = \sup_{\sigma>\kappa} \int\limits_{-\infty}^{\infty} (1+|p|^2)^k \|\breve{u}(\xi,p)\|^2_{m,p}\, d\tau < \infty. \tag{6}$$

Equalities (5) and (6) define, respectively, the norms $[\hat{u}]_{m,k,\kappa}$ and $\|\hat{u}\|_{m,k,\kappa}$ on $\mathcal{H}^{\mathcal{L}}_{m,k,\kappa}(\mathbb{R}^2)$ and $H^{\mathcal{L}}_{m,k,\kappa}(\mathbb{R}^2)$.

Let $\mathcal{H}^{\mathcal{L}^{-1}}_{m,k,\kappa}(G)$ and $H^{\mathcal{L}^{-1}}_{m,k,\kappa}(G)$ be the spaces of the inverse Laplace transforms $u(x,t)$ of $\hat{u} \in \mathcal{H}^{\mathcal{L}}_{m,k,\kappa}(\mathbb{R}^2)$ and $\hat{u} \in H^{\mathcal{L}}_{m,k,\kappa}(\mathbb{R}^2)$, with norms $[u]_{m,k,\kappa;G} = [\hat{u}]_{m,k,\kappa}$ and $\|u\|_{m,k,\kappa;G} = \|\hat{u}\|_{m,k,\kappa}$, let $\mathcal{H}^{\mathcal{L}^{-1}}_{m;k,l;\kappa}(G) = \mathcal{H}^{\mathcal{L}^{-1}}_{m,k,\kappa}(G) \times \mathcal{H}^{\mathcal{L}^{-1}}_{m,l,\kappa}(G)$, where $m,k,l \in \mathbb{R}$, be the space of all $U = (u^{\mathrm{T}}, u_4)^{\mathrm{T}}$, $u = (u_1,u_2,u_3)^{\mathrm{T}}$, with norm $[U]_{m;k,l;\kappa;G} = [u]_{m,k,\kappa;G} + [u_4]_{m,l,\kappa;G}$, and let $H^{\mathcal{L}^{-1}}_{m;k,l;\kappa}(G) = H^{\mathcal{L}^{-1}}_{m,k,\kappa}(G) \times H^{\mathcal{L}^{-1}}_{m,l,\kappa}(G)$, $m,k,l \in \mathbb{R}$, be equipped with the norm $\|U\|_{m;k,l;\kappa;G} = \|u\|_{m,k,\kappa;G} + \|u_4\|_{m,l,\kappa;G}$. We write $H^{\mathcal{L}^{-1}}_{1;0,0;\kappa}(G) = \mathbb{H}^{\mathcal{L}^{-1}}_{1,\kappa}(G)$ and $\|U\|_{1;0,0,\kappa;G} = \|U\|_{1,\kappa;G}$. It is clear that $\mathbb{H}^{\mathcal{L}^{-1}}_{1,\kappa}(G)$ is the subspace of $\mathbb{H}_{1,\kappa}(G)$ consisting of all $U = (u^{\mathrm{T}}, u_4)^{\mathrm{T}}$ such that $\gamma_0 U = 0$.

Theorem 4. *For any $\mathcal{Q} \in \mathcal{H}^{\mathcal{L}^{-1}}_{-1;1,1;\kappa}(G)$, $\kappa > 0$, equation* (4) *has a unique solution $U \in \mathbb{H}^{\mathcal{L}^{-1}}_{1,\kappa}(G)$. If $\mathcal{Q} \in \mathcal{H}^{\mathcal{L}^{-1}}_{-1;k,k;\kappa}(G)$, then $U \in H^{\mathcal{L}^{-1}}_{1;k-1,k-1;\kappa}(G)$ and*

$$\|U\|_{1;k-1,k-1,\kappa;G} \le c[\mathcal{Q}]_{-1;k,k,\kappa;G}.$$

The above assertions are proved by investigating the mapping properties of the operators defined by the plate potentials in the appropriate spaces.

The corresponding results without thermal effects were obtained in [3].

References

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