Multiwavelet Collocation Boundary Element Solution of Laplace's Equation

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Summary

In this paper we present the result of a preliminary investigation into the collocation method, based on multiwavelet approximation, for boundary integral equations with *analytically standard* kernels. We apply the method to the boundary integral solution of Laplace's equation exterior to a closed curve. We present some numerical results which show a good level of compression.

Multiwavelets on [0,1]

Wavelets are generated by a *mother* wavelet $\psi(x)$ from which all other wavelets are obtained using the definition $\psi_\lambda(x) := 2^{\frac{m}{2}}\psi(2^m x - l) \; \forall \; \lambda := \{m,l\} : \; m,l \in \mathbb{Z}$, see [1]. They have proved to be efficient and effective bases for function approximations, as the coefficients of a wavelet expansion decay rapidly for a large class of functions. Due to the multiresolution property of wavelets they provide accurate local descriptions of functions efficiently. For example in the presence of corners and edges, the functions can still be approximated with a linear combination of just a few wavelet bases.

Wavelets are attractive for the numerical solution of integral equations, because their vanishing moments property leads to operator compression [2], [3]. However, to obtain wavelets with compact support and high order of vanishing moments, the length of the support increases as the order of the vanishing moments increases. This causes difficulties with the practical use of wavelets particulary at edges and corners. With multiwavelets, an increase in the order of vanishing moments is obtained not by increasing the support but by increasing the number of *mother* wavelets.

Suppose k is a positive integer and m a non-negative integer, we define the space V_m^k of piecewise polynomial functions

$$V_m^k := \left\{ \begin{array}{c} f : f|_{[2^{-m}n, 2^{-m}(n+1)]} \text{ is a polynomial of degree less than } k \\ \forall \, n = 0, 1, \ldots, 2^m - 1 \text{ and vanishes elsewhere} \end{array} \right\}.$$

It is clear that $V_0^k \subset V_1^k \subset \ldots \subset V_m^k \subset \ldots \subset L_2[0,1]$. For $m = 0, 1, 2, \ldots$, we define the space W_m^k to be the orthogonal complement of V_m^k in V_{m+1}^k; that is $V_{m+1}^k = V_m^k \oplus W_m^k$. Then we have the decomposition $V_m^k = V_0^k \oplus W_0^k \oplus W_1^k \oplus \ldots \oplus W_{m-1}^k$.

The space V_0^k is the space of polynomials of degree less than k on the interval [0,1] and we assume $\{\phi_1, \phi_2, \ldots, \phi_k\}$ to be a basis for it. These are known as the *scaling functions*. Suppose $\{\psi_1, \psi_2, \ldots, \psi_k\}$ is a basis of W_0^k. Therefore, for the orthogonality condition

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$V_0^k \perp W_0^k$ to be satisfied we require the first k moments of $\{\psi_1, \ldots, \psi_k\}$ to vanish. That is $\int_0^1 \psi_j(x) x^i dx = 0$ for $j = 1, 2, \ldots, k$; $i = 0, 1, \ldots, k-1$. The basis functions $\{\psi_1, \psi_2, \ldots, \psi_k\}$ are known as *mother wavelets*.

The $2k$-dimensional space W_1^k is spanned by the functions $\{\psi_1(2x), \ldots, \psi_k(2x), \psi_1(2x - 1), \ldots, \psi_k(2x - 1)\}$. In a similar manner we can obtain the finer $2^m k$-dimensional space W_m^k from the space W_{m-1}^k. To introduce a more convenient notation, if we define $\psi_\lambda := 2^{\frac{m}{2}} \psi_j(2^m x - l)$, where $\lambda := \{j, m, l\}$, the space W_m^k is spanned by the collection

$$\Psi_m := \{\psi_\lambda : l = 0, \ldots, 2^m - 1, \ j = 1, \ldots, k\}. \tag{1}$$

Therefore, the wavelet spaces $\{W_m^k\}$ are generated from the k mother wavelets $\{\psi_1, \psi_2, \ldots, \psi_k\}$. Similarly the spaces $\{V_m^k\}$ can be generated from the scaling functions $\{\phi_1, \phi_2, \ldots, \phi_k\}$, as the span of the collection

$$\Phi_m := \left\{\phi_\lambda = 2^{\frac{m}{2}} \phi_j(2^m x - l) : l = 0, \ldots, 2^m - 1, \ j = 1, \ldots, k\right\}. \tag{2}$$

Let us now define the collection of basis functions Ψ^M as follows:

$$\Psi^M := \Phi_0 \bigcup_{m=0}^{M-1} \Psi_m. \tag{3}$$

It is easy to see that both Φ_M and Ψ^M are bases for the $2^M k$-dimensional space V_M^k, the largest subspace of $L_2[0, 1]$ which we use in a given numerical experiment. It is the collection of wavelet basis functions Ψ^M, given by (3), that we use to obtain operator compression. By estimating the size of the matrix elements, we are able to decide a priori which elements are going too small not to affect the accuracy of our approximation. This way we avoid computing them in the first place, resulting in a fast algorithm with computational cost $O(n \ln^d n)$.

Our approach is in contrast to much of that in current use, where, because of the perceived and real complexity in directly using wavelet bases, many practitioners obtain the matrix compression offered by wavelets by adopting a two stage scheme. First, the standard boundary element matrix is computed using the scaling function bases Φ_M for V_M^k. Then, a wavelet transform is applied to obtain the coefficient matrix with respect to the wavelet basis (3). The resulting matrix is then compressed by the application of a threshold, see [4]. Whilst this method results in some speed up of the solution time, its computation cost is still $O(n^2)$.

We derive the *mother wavelets* ψ_j by using a procedure described in [5]. This makes use of the Gram-Schmidt orthogonalisation process. We consider the cases when $k = 2, 3, 4$.

Usually when implementing a wavelet basis the Galerkin projection method is used; see [6] and references therein. The vanishing moments property is applied twice, leading to substantial matrix compression. However, the Galerkin method requires double integrations in each direction, rather than the single integrations that the collocation method

requires, causing difficulties with efficient implementation of this method; see [7] where results of a comparison between the two methods are presented. Therefore we wish to investigate the implementation of the multiwavelet basis with the collocation method.

Matrix Bounds

The collocation method used with multiwavelet basis functions results in coefficient matrices that are numerically sparse. Therefore we wish to know the size and position of the "small" matrix elements a priori, so that we do not have to compute them. To this end we find a bound for the size of the matrix elements.

Consider matrix elements of the form

$$A_{i,\lambda} = \int_{I_\lambda} \overline{\mathcal{K}}(x_i, y)\, \psi_\lambda(y)dy, \tag{4}$$

where x_i are the collocation points with $i = 1, 2, ..., k2^M$ and $\psi_\lambda \in \Psi^M$. The kernel $\overline{\mathcal{K}}(x, y)$ is the so-called transformed kernel, when the integration domain is changed from Γ to $[0, 1]$. The kernels in many boundary integral equations fall into the class of *analytically standard* functions. The kernel, $\mathcal{K}(x, y)$, is called *analytically standard* of order $2q$ if the transformed kernel, $\overline{\mathcal{K}}(x, y)$, satisfies

$$\left| \partial_x^\alpha \partial_y^\beta \overline{\mathcal{K}}(x, y) \right| \leq D \frac{(|\alpha| + |\beta|)!}{dist(\widehat{x}, \widehat{y})^{1+|\alpha|+|\beta|+2q}} \tag{5}$$

where

$$\overline{\mathcal{K}}(x, y) := \mathcal{K}(\kappa(x), \kappa(y)) |\kappa_x| |\kappa_y| \tag{6}$$

with $\widehat{x} := \kappa(x)$ and $\widehat{y} := \kappa(y)$ and $|\kappa_x|, |\kappa_y|$ are the Jacobians for the parametric map κ, [8].

First we rewrite our analytically standard kernel in (4), as a $(k+1)$-term Taylor expansion about the point (x_i, y_0), with y_0 taken as the midpoint of I_λ. Therefore, we obtain

$$|A_{i,\lambda}| = \left| \int_{I_\lambda} \left[\overline{\mathcal{K}}(x_i, y_0) + \left.\frac{\partial \overline{\mathcal{K}}}{\partial y}\right|_{y=y_0} (y - y_0) + ... + \left.\frac{\partial^k \overline{\mathcal{K}}}{\partial y^k}\right|_{y=t_y} \frac{(y - y_0)^k}{k!} \right] \psi_\lambda(y)dy \right|. \tag{7}$$

Applying the vanishing moments property of the multiwavelets to expansion (7), the first k terms are zero. We then apply inequality (5) to the last term of (7) to obtain the following bound for size of the matrix elements,

$$|A_{i,\lambda}| \leq D \frac{2^{-m(k+\frac{1}{2})-k}}{(2k+1)dist(\widehat{x}_i, \Gamma_\lambda)^{1+k+2q}}. \tag{8}$$

We can now use this bound to decide a priori the position of small matrix elements, that is matrix elements below the level of discretisation error. Therefore we only compute and store the $O(n \ln^d n)$ significant elements.

Boundary Integral Equation

In this paper we consider the solution of Laplace's equation $\nabla^2 u = 0$ in the exterior domain D_+. With a careful application of Green's second theorem, using the condition $|u| = 0$ at infinity, we can obtain the integral representation of the solution to our problem in the form:

$$u(\mathbf{p}) = \int_\Gamma u(\mathbf{q}) \frac{\partial G(\mathbf{p}, \mathbf{q})}{\partial n_{\mathbf{q}}} \, d\Gamma(\mathbf{q}) - \int_\Gamma G(\mathbf{p}, \mathbf{q}) \frac{\partial u}{\partial n_{\mathbf{q}}}(\mathbf{q}) \, d\Gamma(\mathbf{q}), \quad \mathbf{p} \in D_+, \tag{9}$$

where $G(\mathbf{p}, \mathbf{q}) = \frac{-1}{2\pi} \ln|\mathbf{p} - \mathbf{q}|$ is the free space Green's function (fundamental solution) of Laplace's equation. This representation requires both the Dirichlet and the Neumann data, the so-called "Cauchy Data".

Letting $p \in D_+ \to p \in \Gamma$ and using the continuity and jump discontinuity properties of the single layer potential ($\mathcal{L}$) and the double layer potential ($\mathcal{M}$) operators respectively, we obtain

$$(-\frac{1}{2} I + \mathcal{M}) u = \mathcal{L} \frac{\partial u}{\partial n}, \quad \mathbf{p} \in \Gamma. \tag{10}$$

We consider the case of the Neumann boundary condition where $\frac{\partial u}{\partial n}|_\Gamma$ is given. We solve the boundary integral equation (10) to find the missing Dirichlet data.

Numerical Results

In this section we present numerical results for the Laplace problem exterior to an elongated ellipse of circumference 4π, with major axis 2.9297628 and minor axis 0.7324407, centered at the origin. We consider a Neumann problem, equivalent to that generated by the point source technique when the source point is placed at $\mathbf{p}_0 = (2.878567, \frac{27}{100}\pi)$ and with strength 1.3. That is, the field generated is $u(\mathbf{p}) = -\frac{1.3}{2\pi} \ln|\mathbf{p} - \mathbf{p}_0|$. In tables 1-3, $\|u - u_h\|$ is the L_2 norm of the error and nz is the number of non-zero elements remaining after compressing the matrix. The compression column gives the percentage of the matrix entries that can be set to zero without any detrimental effect. Our matrices are of size $k2^M$. Note that as the number of elements increases so does the level of compression.

Table 1: k=2

M	$\|f - f_h\|$	nz	compression %
3	3.95×10^{-2}	184	28.2
4	3.09×10^{-2}	500	51.2
5	1.24×10^{-2}	1344	67.2
6	4.86×10^{-3}	3960	75.8
7	8.03×10^{-4}	13976	78.7
8	2.73×10^{-4}	39246	84.9
9	7.00×10^{-5}	121516	88.4
10	1.76×10^{-5}	425560	89.9

Table 2: k=3

M	$\|f - f_h\|$	nz	compression %
3	4.93×10^{-2}	334	42.0
4	1.81×10^{-2}	954	58.6
5	6.10×10^{-3}	2440	73.5
6	1.42×10^{-3}	6718	81.7
7	2.91×10^{-4}	19458	86.8
8	3.52×10^{-5}	58742	90.0
9	4.21×10^{-6}	185808	92.1
10	5.30×10^{-7}	588192	93.8

Table 3: k=4

M	$\|f - f_h\|$	nz	compression %
3	3.31×10^{-2}	554	45.9
4	2.19×10^{-2}	1422	65.3
5	2.96×10^{-3}	4032	75.4
6	8.69×10^{-4}	10050	84.7
7	4.62×10^{-5}	29592	88.7
8	3.84×10^{-6}	87020	91.7
9	3.21×10^{-7}	268312	93.6
10	2.02×10^{-8}	858132	94.9

For each of the cases considered we observe $O(h^k)$ convergence. This is the predicted order from the theory of convergence of collocation methods in Sobolev spaces, given by:

$$\| u - u_h \|_{\mathscr{H}^r} \leq Ch^{(t-r)} \| u \|_{\mathscr{H}^t}, \tag{11}$$

where $r = 0$ and $t = k$ are taken here. We note that the larger the value of k is, the larger the level of compression will be.

Conclusion

An efficient numerical algorithm for the solution of a large class of boundary integral equations has been presented based on collocation methods using multiwavelet bases.

We have applied the method to the solution of the Laplace's equation in an exterior domain. Numerical results have been presented demonstrating the level of matrix compression achieved in line with that predicted by the theory.

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