Hybrid numerical method for transient waves in laminated cylindrical shells

X. Han[1,2*], G.Y. Li[1] and G.R. Liu[1,2]

Summary

An efficient method is presented for investigating transient response of axisymmetric cross-ply laminated cylindrical shells subjected to an impact load. In this hybrid numerical method, the laminated cylindrical shell is divided into layered cylindrical elements in the thickness direction. The Hamilton principle is used to develop governing equations of the shell. The displacement response is determined by employing the Fourier transformations and the modal analysis. The method is applied to analyze transient waves in axisymmetric laminated cylindrical shells, and its efficiency is demonstrated.

Introduction

Composite cylindrical shells have high strength and stiffness, low coefficient of thermal expansion and excellent corrosion resistance. Besides, they also possess an ease of fabrication, a relative low-cost process and well-adapted production. In the application of laminated cylindrical structures to aerospace, nuclear and automobile industries, analyses of transient waves in them are of very importance.

There have been a number of research works on wave propagation problems related to composite cylindrical shells [1-5]. Recently Han et al. dealt with transient responses [6] of cylindrical shells made of functionally gradient materials, and characteristics of waves [7] in a functionally graded cylindrical shell. The hybrid numerical method [4-5] is one of the most efficient numerical tools for 2D or 3D transient wave analysis in composite laminated plates. The efficiency is achieved by combining the finite element technique with the numerical Fourier transformation technique as well as the model analysis technique for dealing with the time integration.

This paper presents a hybrid numerical method for calculating the transient response of laminated cylindrical shells. In this method, the laminated cylindrical shell is divided into layered cylindrical elements in the thickness direction. The Hamilton principle is used to develop governing equations of the cylinder. The displacement response is determined by employing the Fourier transformations and the modal analysis. This hybrid numerical method can reduce the number of elements, thus it has outstanding computation efficiency in the transient wave analysis of laminated cylindrical structures.

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[1] College of Mechanical and Automotive Engineering, Hunan University, Changsha 410082, Hunan, P. R. China
[2] Centre for Advanced Computations in Engineering Science (ACES), Department of Mechanical Engineering, National University of Singapore, 10 Kent Ridge Crescent, Singapore 119260
* Corresponding author, E-mail: acehanxu@nus.edu.sg

Basic Equation

Consider a laminated composite circular cylindrical shell made of an arbitrary number of linearly elastic cross-ply shell-like plies. The inner radius and outer radius of the cylinder are denoted by R_1, R_2 respectively, as shown in Fig. 1. Let x and z denote respectively the axial and radial coordinates. The cylinder is subjected to a radial line load of $f(t)$ uniformly distributed along the circumferential direction. Because the geometry of the cylinder and the load are independent of the circumferential direction, the problem is axisymmetric.
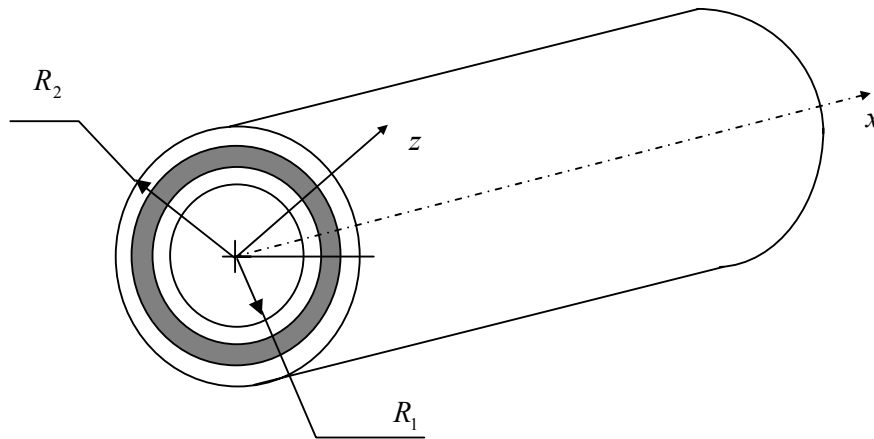

Figure 1 A cross-ply laminated circular cylindrical shell.

The stain-displacement relations are given by

$$\boldsymbol{\varepsilon} = \mathbf{L}\mathbf{U} \tag{1}$$

where $\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x & \varepsilon_\theta & \varepsilon_z & \varepsilon_{xz} \end{bmatrix}^T$ is the vector of stains, $\mathbf{U} = [u\ w]^T$ is the vector of displacements, u and w are the displacements in the axial and radial directions, respectively, and $\mathbf{L}$ is the operator matrix given by

$$\mathbf{L} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & 0 & \dfrac{\partial}{\partial z} \\ 0 & \dfrac{1}{R_1 + z} & \dfrac{\partial}{\partial z} & \dfrac{\partial}{\partial x} \end{bmatrix}^T \tag{2}$$

The operator matrix $\mathbf{L}$ can be rewritten as

$$\mathbf{L} = \mathbf{L}_1 \frac{\partial}{\partial x} + \mathbf{L}_2 \frac{\partial}{\partial z} + \mathbf{L}_3 \frac{1}{R_1 + z} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial x} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial z} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{R_1 + z} \quad (3)$$

where the radial coordinate is decomposed as $R_1 + z$. The stresses are related to strains by

$$\boldsymbol{\sigma} = \mathbf{c}\boldsymbol{\varepsilon} \quad (4)$$

where $\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \sigma_\theta & \sigma_z & \sigma_{xz} \end{bmatrix}^T$ is the vector of stresses and

$$\mathbf{c} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{15} \\ & c_{22} & c_{23} & c_{25} \\ & & c_{33} & c_{35} \\ s\text{ym} & & & c_{55} \end{bmatrix} \quad (5)$$

is the matrix of the elastic constants of the lamina whose expressions in terms of engineering constants are given in detail by Vinson and Sierakowski [8].

Dispersion equation

With the above basic equations, we apply the Hamilton's principle to derive the dispersion equation for the shell. In this study, we first use finite elements to model the radial displacement of the shell and then employ the complex exponentials to model the axial and circumferential displacements. In view of the heterogeneity of the laminated composite shell in the radial direction, an annular element shown in Figure 2 is used in the subdivision of the shell so as to achieve high computational precision. The thickness and inner radius of the nth element are denoted by h_n and r_n, respectively. It can be found that the outer radius of the nth element is equal to $r_n + h_n$.

Approximating the displacement field within an element as

$$\mathbf{U} = \mathbf{Nd} \quad (6)$$

where $\mathbf{N}$ is the shape function matrix of second order given by

$$\mathbf{N} = [(1 - 3\bar{z} + 2\bar{z}^2)\mathbf{E} \quad 4(\bar{z} - \bar{z}^2)\mathbf{E} \quad (2\bar{z}^2 - \bar{z})\mathbf{E}] \quad (7)$$

Here $\mathbf{E}$ is a 2 by 2 identity matrix and $\bar{z} = z/h_n$, and $\mathbf{d}$ is the displacement amplitude vector at $z = 0, z = 0.5h_n$ and $z = h_n$ as follows:

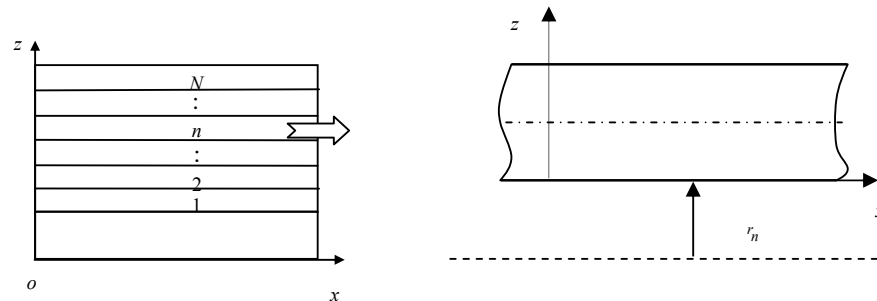$$\mathbf{d} = [d_l^T \ d_m^T \ d_n^T]^T \quad (8)$$

Figure 2 Annular element subdivision

The potential energy of the element in the absence of body force is given by

$$V = \pi \int_0^{h_n} \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}(r+z)dz - 2\pi wqR_2 \tag{9}$$

The kinetic energy of the element is expressed in terms of the displacement vector as

$$T = \pi \int_0^{h_n} \frac{\partial \mathbf{U}^T}{\partial t} \frac{\partial \mathbf{U}}{\partial t} \rho(r+z)dz \tag{10}$$

Applying the Hamilton's principle and assembling all the elements, a system of approximate differential equations for the whole cylinder is expressed as

$$\mathbf{q}_t = \mathbf{M}_t \ddot{\mathbf{d}}_t + \mathbf{K}_{\mathbf{D}_t} \mathbf{d}_t \tag{11}$$

where

$$\mathbf{K}_{\mathbf{D}_t} = -\mathbf{A}_{2t} \frac{d^2}{dx^2} + \mathbf{A}_{1t} \frac{d}{dx} + \mathbf{A}_{0t} \tag{12}$$

In these equations, the subscript 't' denotes matrices or vectors for the whole cylinder. The matrices $\mathbf{A}_{it}(i=0,1,2)$, $\mathbf{M}_t$ and the vectors $\mathbf{q}_t$, $\mathbf{d}_t$ can be obtained by assembling the corresponding matrices and vectors of adjacent elements.

Transient response

We introduce the Fourier transformations with respect to the axial coordinate x to the equation (11) leads to

$$\widetilde{\mathbf{q}}_t = \mathbf{M}_t \ddot{\widetilde{\mathbf{d}}}_t + \mathbf{K}_t \widetilde{\mathbf{d}}_t \tag{13}$$

where $\widetilde{\mathbf{d}}$ and $\widetilde{\mathbf{q}}_{\mathbf{t}}$ are the Fourier transformations of $\mathbf{d}$ and $\mathbf{q_t}$, respectively, and $\mathbf{K}_t$ is the stiffness matrix given by

$$\mathbf{K}_t = k^2\mathbf{A}_{2t} + ik\mathbf{A}_{1t} + \mathbf{A}_{0t} \tag{14}$$

The modal analysis is used to obtain the Fourier transformation of the displacement vector. Solving the eigenvalue equation corresponding to equation (13), we can give the displacement in the Fourier transformation domain. Taking the inverse Fourier transformation, the displacement response in the space-time domain can be expressed by

$$\mathbf{d}_t(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\widetilde{\mathbf{d}}_t(k,t)e^{-ikx}dk \tag{15}$$

The integration in equation (15) [4] can be carried out by using the fast Fourier transform (FFT) techniques.

In the present study, the time history of the incident wavelet is given as

$$f(t) = \begin{cases} \sin(2\pi t/t_d) & 0 < t < t_d \\ 0 & t \le 0 \text{ and } t \ge t_d \end{cases} \tag{16}$$

where t_d is the time duration of the incident wavelet and $\omega_f = \dfrac{2\pi}{t_d}$.

Numerical results and Conclusions

In this section, numerical results are presented for transient response of impact loads in a glass-carbon laminated composite cylindrical shell. The on-principal-axis elastic constants of the shell are taken from Takahashi and Chou [9]. In the calculations, the dimensionless parameters introduced in [6] are used herein.
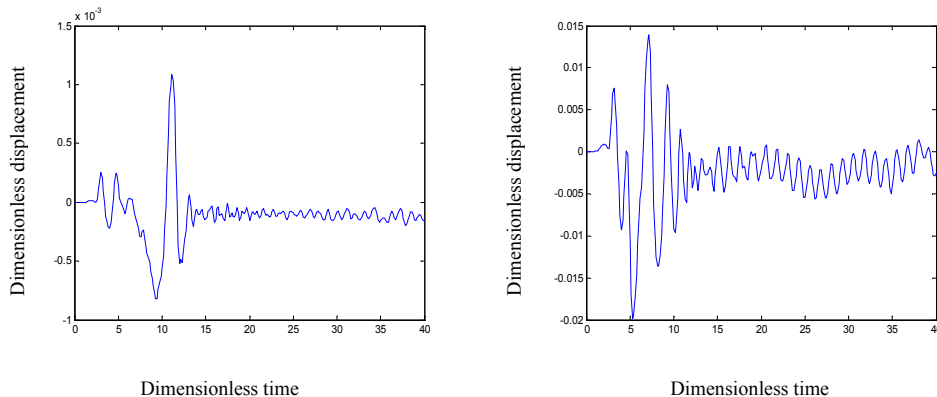
(a) $R_2 = 20H$ b) $R_2 = H$

Figure 3 The time history of the displacement w at $x = 10H$ on the upper surface

Figures 3 (a) and (b) show the time history of the axial and radial displacements on the outer surface of a glass-carbon $(C/G/C)_s$ cylindrical shell ($R_2 = 20H$) subjected to a radial line load defined by equation (16) uniformly distributed along the circumferential direction at $x = 0$, respectively.

This presented HNM method is very effective to investigate the transient response in laminated cylindrical shells, the running time for 12 sub-layers cylinder only takes about 30 seconds. The efficiency is achieved by combining the finite element technique with the numerical Fourier transformation technique as well as the model analysis technique for dealing with the time integration.

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