On Convergence of the Series Solution for the Crack Propagation in Stochastically Inhomogeneous Materials

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Summary

The present work is a continuation of a research on a crack propagation in stochastically inhomogeneous materials presented at earlier ICCES conferences [1, 2]. The ESIF (effective stress intensity factor), and the initial direction of the crack were investigated in [1,2,4] based on the series solution. The radius of convergence of the series representing the ESIF in stochastically inhomogeneous material is considered. There is a number of theoretical models which have been developed to predict the effective elastic moduli of composite materials (see, for example, review [3]). However, the effective elastic moduli depend on the shape of the body. Since a crack forms a new boundary, a corresponding boundary value problem must be considered.

Introduction

In stochastically inhomogeneous solids with macrocracks, the effective elastic moduli become functions of the geometric parameters of the cracks, particularly, their lengths. The boundary value problem for the macrocrack in stochastically inhomogeneous material is considered. It was shown in [4] that in the frame of the linear elasticity means of the stresses in the vicinity of the radius r of the crack still have a well known order $(r)^{-1/2}$ of singularity which justifies the introduction of the "effective stress intensity factors (ESIF)" for stochastically inhomogeneous materials with cracks. Several terms of the series solution for the ESIF were obtained in [1,2]. To find the radius of convergence of this series, the following problem is considered.

Statement of the Problem

The closed system of the equations of the plane elasticity in heterogeneous body has a well known form:

$$\sigma_{ij,j} = 0; \quad \Delta(\gamma\sigma_{mm}) = \sigma_{ij}\frac{\partial^2 q}{\partial x_i \partial x_j}; \quad \sigma_{ij} n_j = g_i \Big|_L;$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}; \quad \gamma = \frac{1}{E}; \quad q = \frac{(1+\nu)}{E}$$

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where σ_{ij} are stresses, g_j are known deterministic functions on the boundary L, and elastic constants q and γ are introduced as functions of the Young modulus E and the Poisson ratio ν. Let $q(x,y)$ and $\gamma(x,y)$ be random functions of coordinates. Then the problem (1) is statistically nonlinear with respect to the random functions σ_{ij}. Problem (1) can be linearized if we assume that functions $q(x,y)$ and $\gamma(x,y)$ are statistically homogeneous, that is, their means $\langle q \rangle = const$; $\langle \gamma \rangle = const$, and can be represented as sums of their means and perturbations. We seek a solution to the problem as a power series over the parameter λ in the following form

$$q = \langle q \rangle + \lambda q'; \quad \gamma = \langle \gamma \rangle + \lambda \gamma'; \quad \sigma_{ij} = \sum_{k=0}^{\infty} \lambda^k \sigma_{ij}^{(k)} \qquad (2)$$

Series Solution

Introducing complex variable z, using methods of [5], and substituting (2) into (1), and equating factors of the same powers of λ, we can obtain the boundary value problem for initial approximation

$$\langle \gamma \rangle \Delta(\sigma_x^0 + \sigma_y^0) = 0; \text{ (compatibility conditions)}$$

$$(X_n^0 + Y_n^0)_L \Big| = g_1 + ig_2; \text{(boundary conditions)} \qquad (3)$$

$$\frac{\partial}{\partial z}(\sigma_x^0 - \sigma_y^0 + 2i\sigma_{xy}^0) + \frac{\partial}{\partial z}(\sigma_x^0 + \sigma_y^0) = 0; \text{ (equilibrium conditions)}$$

and recurrence sequence of statistically linear boundary value problems for successive approximations of the stresses

$$\frac{\partial}{\partial z}(\sigma_x^{(k)} - \sigma_y^{(k)} + 2i\sigma_{xy}^{(k)}) + \frac{\partial}{\partial \bar{z}}(\sigma_x^{(k)} + \sigma_y^{(k)}) = 0; \text{(equilibrium conditions)}$$

$$\langle \gamma \rangle \Delta(\sigma_x^{(k)} + \sigma_y^{(k)}) = f^{(k-1)}; \quad \text{(compatibility conditions)} \qquad (4)$$

$(X_n^{(k)} + Y_n^{(k)})_L \Big| = 0;$ (boundary conditions), where expressions $f^{(k-1)}$ consist of the sums of products of the following type $\sigma_{ij}^{(k-1)} \partial^2 q / \partial z^2$.

The boundary value problem (3) is a problem for the homogeneous solid with a crack loaded with a given stresses, while recurrence relations (4) are the boundary value problems for a crack free of tractions in a solid with distributed body forces. Let us consider the infinite plane (in case of the plane stress), or an infinite cylindrical body (for the plane strain) with a crack of the length $2l$ whose banks are loaded with the given forces. Using the Airy function $U(x, y)$, the problem (4) can be reduced to a biharmonic inhomogeneous equation for each approximation

$$\begin{aligned} &\sigma_x^{(k)} = \frac{\partial^2 U^{(k)}}{\partial y^2}; \quad \sigma_y^{(k)} = \frac{\partial^2 U^{(k)}}{\partial x^2}; \quad \sigma_{xy}^{(k)} = -\frac{\partial^2 U^{(k)}}{\partial x \partial y}; \\ &\Delta^2 U^{(k)} = f^{(k-1)}; \quad (X_n^{(k)} + iY_n^{(k)})\Big|_L = 0 \end{aligned} \tag{5}$$

Solution to the problem (5) was obtained as a sum of solutions to two sub-problems: (I) the inhomogeneous problem for the infinite body without the crack, and (II) homogeneous problem for the infinite body with a cut loaded with stresses opposite to those acting on the cut trace from the problem (I). The problem (I) was solved using the Green's function for the biharmonic equation for the infinite plane. The problem (II) for the infinite plane with a linear cut on $[-l, l]$ can be written in the following form

$$\Delta^2 U^{(k)} = 0; \quad (X_n^{(k)} + iY_n^{(k)})\Big|_L = -(X_{n1}^{(k)} + iY_{n1}^{(k)})\Big|_L;$$

where $(X_n^{(k)}, Y_n^{(k)})$ and $(X_{n1}^{(k)}, Y_{n1}^{(k)})$ represent, correspondingly, resultant vectors of the forces applied to the cut line, obtained from the solution of the problem (I), was solved by the method of complex potentials [4]. The complex potentials for the k–th approximation were found in the following form

$$\Phi^{(k)}(z) = -\frac{1}{2\pi}\int_{-l}^{l}\frac{\sqrt{t^2 - l^2}\,(\sigma_{y1}^{(k)} - \sigma_{xy1}^{(k)})dt}{t - z}$$

where $(\sigma_{y1}^{(k)} - \sigma_{xy1}^{(k)})$ are stresses on the trace of the cut that was found by the Airy function as a solution of the non-homogeneous biharmonic equation. It was shown by the asymptotic analysis that means of the complex potentials preserved their ordered of singularity of $(r)^{-1/2}$ at the tips of the crack, which justifies the introduction of the effective stress intensity factor (SIF) in the non-homogeneous solid with a crack whose k–th approximation has a form

$$\left\langle K^{(k)}\right\rangle = \left\langle K_I^{(k)} - iK_{II}^{(k)}\right\rangle =$$
$$\frac{i}{8\pi\sqrt{l}}\int_{-l}^{l}\iint_S \sqrt{\frac{t+l}{t-l}}\left\langle f^{(k-1)}(\xi,\eta)\right\rangle\left\{\frac{(t-\xi)^2-\eta^2}{(t-\xi)^2+\eta^2}+\ln\left|(t-\xi)^2+\eta^2\right|\right\}d\xi d\eta dt \quad (6)$$

The statistical characteristics of the material's elastic moduli are involved into expression $\left\langle f^{(k-1)}\right\rangle$. The further analysis depends on the correlation function of the distribution of inclusions and their shapes. Using the indicator function, the elastic compliances can be introduces in the following form

$$\begin{aligned}\gamma(\rho) &= \gamma_1\chi(\rho)+\gamma_2[1-\chi(\rho)]\\ q(\rho) &= q_1\chi(\rho)+q_2[1-\chi(\rho)]\end{aligned} \quad (7)$$

for inclusions or $E(\rho)=E_1\chi(\rho)+E_2[1-\chi(\rho)]$ for microvoids or microcracks,

with mean of the indicator function $\left\langle \chi\right\rangle = \omega\dfrac{N\alpha^2}{S} = n\dfrac{\alpha^2}{l^2}$ (8)

where ω is the average density of microdefects, n is an average number of microdefects in a l x l square, and N is the total number of microdefects in a body cross-section S.

The variances of the elastic compliances can be introduced as follows

$$\begin{aligned}&V_\gamma = (\gamma_1-\gamma_2)^2\omega(1-\omega); \quad V_q = (q_1-q_2)^2\omega(1-\omega);\\ &\left\langle\gamma\right\rangle = \gamma_2+(\gamma_1-\gamma_2)\omega; \quad \left\langle q\right\rangle = q_2+(q_1-q_2)\omega;\end{aligned} \quad (9)$$

By substituting (9) into (6), the n-th approximation of mean of ESIF was evaluated by *Mathematica*. It was shown that statistical moments of odd order are equal to zero. In the frame of the correlation theory, the even order moments can be introduced through the correlation function if ordinates of the fluctuations of compliances be assumed normally distributed. It was shown that a parameter [6]

$$c = \frac{|E_1(1+\nu_2)-E_2(1+\nu_1)|\sqrt{\omega(1-\omega)}}{E_1(1+\nu_2)\omega+E_2(1+\nu_1)(1-\omega)} < 0.89 \quad (10)$$

governs a convergence of the n-th approximation of mean of ESIF, and that the fourth approximation of $\left\langle K^{(4)}\right\rangle$ differs very insignificantly significantly from

$\left\langle K^{(2)} \right\rangle$ which coincides with the result obtained for a crack interacting with a discrete field of microcracks [7].

Conclusion

Suggested approach allows to obtain an approximation of the ESIF, an initial angles and the critical loads of the crack propagation at the tip of the crack in various composite materials, as soon as correlation functions of elastic moduli are known. The present work puts a boundary on an applicability of a small parameter method for a crack propagation in a stochastically inhomogeneous material according to (10).

Another approach to a crack modeling in the stochastic inhomogeneous material implies the use of a model that accounts for explicit interaction of the macrocrack with a microcrack, micropore or microinclusion (see review [7]). One of such model was developed in [7], and applied to various problems in [9]. Modeling a crack in non-homogeneous solid and assuming a desired density of microdefects generated randomly, means of the SIF's were found on the basis of n such generated realizations.

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