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On the Numerical Modelling of Orthotropic Large Strain Elastoplasticity

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Summary

A constitutive model for orthotropic yield function at large strain elastoplasticity is described in an invariant setting related to the referent configuration. The invariants are expressed in terms of deviatoric part of Eshelby' stress tensor and structural tensors. The material model enjoys a feature of the multiplicative decomposition of the deformation gradient. Kinematic hardening is combined with isotropic hardening. An accurate and trivial wise objective integration algorithm employing the exponential map is derived. The performance of the proposed formulation is demonstrated by numerical simulation

Introduction

Many materials show anisotropic behaviour owing to their generally orientation dependent structure. The orthotropic material symmetries can be described by structural tensors [1]. R. Hill [2] was the first to establish initial yielding anisotropic criteria and to validate them experimentally. A small strain model based on this concept is presented in [3]. More recently, an anisotropic formulation at large strain elastoplasticity has attracted considerable attention [4], [5], [6], [7]. A lot of numerical algorithms have been developed but they are not without shortcomings. Many questions which have arisen should be solved in future researches.

This paper deals with the numerical modelling of the orthotropic elastoplastic responses at large strains. The constitutive model and orthotropic yield functions are presented. Anisotropic yielding response is described in quadratic invariant form. Invariants are the functions of deviatoric relative stress, in terms of Eshelby like stress tensor, and the structural tensors. The structural tensors represent the privileged directions of the material which are not altered during the deformation process.

The proposed model is formulated in the spatial configuration and then, for numerical convenience, reformulated at the referent configuration. The formulation is based on the multiplicative decomposition of the deformation gradient, isotropic free energy function and orthotropic yield condition formulated in an invariant setting. In addition, an isotropic hardening response and a free energy-based model of the kinematic hardening are also included [8],[9].

The theory and the computational algorithms have been implemented and applied to a shell finite element developed in [10] and [11]. The shell formulation allows for the use of complete three-dimensional constitutive laws. Finally, a numerical example demonstrates the efficiency of the proposed algorithms.

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Kinematics of the Elastic-Inelastic Body

Let $\mathcal{B} \subset \mathbb{R}^3$ define a body. A motion of the body $\mathcal{B}$ is represented by a one-parameter mapping $\varphi : \mathcal{B} \longrightarrow \mathcal{B}_t$, where $t \in \mathbb{R}$ is the time and $\mathcal{B}_t$ is the current configuration at time t. At the reference configuration, every point is associated with the position vector $\mathbf{X} \in \mathcal{B}$ and at the current configuration with $\mathbf{x} \in \mathcal{B}_t$. The point map then gives $\varphi : \varphi(\mathbf{X}) = \mathbf{x}$. The tangent map related to φ is the deformation gradient $\mathbf{F}$ which maps the tangent space $T_X\mathcal{B}$ at the reference configuration to the tangent space $T_x\mathcal{B}_t$ at the actual configuration, $\mathbf{F} := T_X\mathcal{B} \longrightarrow T_x\mathcal{B}_t$. Therefore, the deformation gradient is a two-point tensor.

For the description of inelastic deformation, the well established multiplicative decomposition of the deformation gradient in an elastic part, $\mathbf{F_e}$, and in an inelastic part, $\mathbf{F_p}$, is assumed:

$$\mathbf{F} = \mathbf{F_e}\mathbf{F_p}. \tag{1}$$

For metals, the inelastic part is accompanied by the assumption $\mathbf{F_p} \in SL^+(3,\mathbb{R})$ which reflects the incompressibility of inelastic deformations, where $SL^+(3,\mathbb{R})$ denotes a special linear group with determinants equal to one. The aforementioned decomposition is usually accepted as equivalent to the introduction of an intermediate configuration. We assume that $\mathbf{F_p}$ is well defined by an adequate evolution equation for an appropriately defined material plastic rate.

Constitutive Relations

Elastic behaviour of a body is assumed to be fully characterized by means of a free energy function ψ. This function depends on the measure of elastic strains and on the internal variables which capture certain physical features of the material micro structure and transfer them to the macro level. These internal variables can be of scalar nature as well as of tensorial nature. Accordingly, we assume the existence of a free energy function $\psi(\mathbf{b_e},\mathbf{b_q},Z)$, where $\mathbf{b_e}$ and $\mathbf{b_q}$ are strain-like tensors defined at the actual configuration and Z is the internal variable energy conjugate to the isotropic hardening variable. Herein, $\mathbf{b_e}$ is understood as the elastic deformation tensor and $\mathbf{b_q}$ is an objective tensor defining internal variable energy conjugate to kinematic hardening variable

$$\mathbf{b_q} = \mathbf{F}\mathbf{F_p}^{-1}\mathbf{F}^{-1}. \tag{2}$$

Dissipation inequality defined as local stress power minus the local rate of change of free energy is expressed in the following form

$$\mathcal{D} = \tau : \mathbf{l} - \rho_0 \frac{\partial\psi}{\partial\mathbf{b_e}} : \dot{\mathbf{b}}_\mathbf{e} - \rho_0 \frac{\partial\psi}{\partial\mathbf{b_q}} : \dot{\mathbf{b}}_\mathbf{q} - \rho_0 \frac{\partial\psi}{\partial Z} \cdot \dot{Z} \geq 0. \tag{3}$$

Assuming that equation (3) has to hold for all possible motions, a classical argument of thermodynamic yields the following:

$$\mathcal{D}_r = \gamma : \mathbf{l_p} + Y \cdot \dot{Z} \geq 0\,, \tag{4}$$

with definition

$$\gamma = \tau - \mathbf{q}, \tag{5}$$

where

$$Y = -\rho_0 \frac{\partial \psi}{\partial Z}, \tag{6}$$

$$\tau = 2\rho_0 \frac{\partial \psi}{\partial \mathbf{b_e}} \mathbf{b_e} + \rho_0 \frac{\partial \psi}{\partial \mathbf{b_q}} \mathbf{b_q}^T - \rho_0 \mathbf{b_q}^T \frac{\partial \psi}{\partial \mathbf{b_q}}, \tag{7}$$

$$\mathbf{q} = -\rho_0 \mathbf{b_q}^T \frac{\partial \psi}{\partial \mathbf{b_q}}. \tag{8}$$

In the above relations ρ_0 is the density at reference configuration, γ is the relative stress, and τ is Kirchhof's stress tensor. Y denotes isotropic hardening and $\mathbf{q}$ is back stress. The part of free energy related to the kinematic hardening is assumed to be of the form

$$\psi_q = \frac{1}{2} c \operatorname{tr}\left(\tilde{\mathbf{b}}_\mathbf{q} \tilde{\mathbf{b}}_\mathbf{q}^T\right), \tag{9}$$

where c is the kinematic hardening parameter and $\tilde{\mathbf{b}}_\mathbf{q}$ is defined as

$$\tilde{\mathbf{b}}_\mathbf{q} = \frac{\mathbf{b_q}}{(\det \mathbf{b_q})^{1/3}}. \tag{10}$$

Using the principle of maximum dissipation we come to evolution equations in the following form

$$\mathbf{l_p} = \lambda \frac{\partial \phi}{\partial \gamma}, \tag{11}$$

$$\dot{Z} = \lambda \frac{\partial \phi}{\partial Y}, \tag{12}$$

where $\mathbf{l_p}$ is spatial inelastic rate, λ is plastic multiplier and ϕ is yield function.

In order to simplify numerical computation, the theory is now reformulated in a purely material setting. For that purpose all equations and variables are pulled back to the reference configuration, as follows

$$\Xi = \mathbf{F}^T \tau \mathbf{F}^{-T}, \qquad \Gamma = \mathbf{F}^T \gamma \mathbf{F}^{-T}, \qquad \mathbf{Q} = \mathbf{F}^T \mathbf{q} \mathbf{F}^{-T}, \tag{13}$$

where Ξ defines a quantity, whose spherical part coincides with Eshelby's stress tensor. The value Γ is the material relative stress defined at the reference configuration and $\mathbf{Q}$ is the material back stress. Now, the evolution equations (11) and (12) at the reference configuration take the form

$$\mathbf{L_p} = \lambda \left(\frac{\partial \phi}{\partial \gamma}\right)^T \tag{14}$$

$$\dot{Z} = \lambda \frac{\partial \phi}{\partial Y} \quad , \tag{15}$$

where $\mathbf{L_p}$ is the material inelastic rate.

Orthotropic Yield Criterion

Orthotropic pressure intensive yield condition is considered by the use of isotropic tensor functions. The yield criterion ϕ is formulated in terms of the relative stress and structural tensors. Accordingly, the yield function has the following form

$$\phi = \sqrt{\frac{2}{3}}\sigma^o_{11}\sqrt{\chi} - \sqrt{\frac{2}{3}}(\sigma^o_{11} + Y), \tag{16}$$

where σ^o_{11} is the yield stress in direction 1 and Y denotes the isotropic hardening function defined as

$$Y = -HZ - (\sigma_\infty - \sigma_Y) \cdot (1 - \exp(-\eta Z)) \,. \tag{17}$$

Herein H is a linear isotropic hardening parameter, σ_∞ stands for the saturation yield stress, while η denotes a constitutive parameter quantifying the rate at which the saturation yield stress is attained during loading. χ is a quadratic flow criterion defined as the function of invariants

$$\chi = \alpha_1 I_1^2 + \alpha_2 I_2^2 + \alpha_3 I_3^2 + \alpha_4 I_1 I_2 + \alpha_5 I_1 I_3 + \alpha_6 I_2 I_3 + \alpha_7 I_4 + \alpha_8 I_5 + \alpha_9 I_6, \tag{18}$$

where the following set of invariants defines the integrity basis

$$I_i = \mathrm{tr}\,[{}_i\mathbf{M}\,\mathrm{dev}\,\Gamma], \tag{19}$$

$$I_{i+3} = \mathrm{tr}\,[{}_i\mathbf{M}\,(\mathrm{dev}\,\Gamma)^2], \qquad i = 1,2,3. \tag{20}$$

Material constants $\alpha_1 - \alpha_9$ are defined by the use of Hill's coefficients [2]. They depend directly on six independent yield stresses σ^o_{ab}. Structural tensors ${}_i\mathbf{M}$ in the case of orthotropy describe the orthotropic material symmetry and they are defined by means of three structural tensors as

$${}_i\mathbf{M} = {}_i\mathbf{v} \otimes {}_i\mathbf{v}, \qquad i = 1,2,3 \qquad \text{(no summation)} \tag{21}$$

where $\mathbf{v}$ is the privileged direction of the material in the reference configuration.

Numerical Formulation

The integration of evolution equations is performed by using the well-known predictor-corrector computational strategy. After updating the state variable at time t_n, the trial step is computed by freezing the plastic flow during the time interval ΔT between the times t_n and t_{n+1}. The exponential map is used for updating the variables and it ensures the fulfillment of the incompressibility condition for inelastic deformation. Accordingly, the plastic parts of the deformation gradient at the time step t_{n+1} may be expressed as

$$\mathbf{F_p}^{-1}|_{n+1} = \exp(-\Delta T\,\mathbf{L_p})\mathbf{F_p}^{-1}|_n. \tag{22}$$

By inserting the update relations of the state variables in the yield function, a non-linear scalar equation for the plastic multiplier is obtained. It has to be solved by employing a local Newton's iterative solution procedure, see [9].

The global iteration scheme should be used to solve global finite element equation. In order to ensure quadratic convergence in a close neighbourhood to the exact solution, the algorithmic tangent operator is derived. It is achieved by the linearization of the second Piola-Kirchhoff tensor $\mathbf{S} = \mathbf{C}^{-1}\Xi$, with respect to the right Cauchy-Green deformation tensor $\mathbf{C}$ as follows

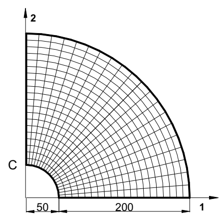
$$\frac{\partial \mathbf{S}}{\partial \mathbf{C}} = \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}}\Xi + \mathbf{C}^{-1}\frac{\partial \Xi}{\partial \mathbf{C}}. \tag{23}$$

Full procedure for return mapping algorithm and the algorithmic tangent operator is shown in [9].

The shell finite element formulation [11] based on a 7-parameter theory including transversal strains is used. Thus, a complete three-dimensional constitutive law may be employed. The finite element is based on a four-node enhanced strain formulation which enables the elimination of the undesired locking phenomena.

Numerical Example

This example will consider elastoplastic deformation of a circular plate under uniform conservative load. The plate is simply supported in direction 3 at the edges so that only horizontal displacements and rotations can occur. Due to the geometrical and material symmetry one quarter of the plate is discretized with 20×20 elements. The material data and geometry of the plate are shown in Fig.1. Computations are performed for the isotropic (A) and orthotropic (B) material. Fig.2 shows deformed configuration for the isotropic case (material A) and for the orthotropy (material B). As expected, for material B the plastic



Material data:
$E = 210$ GPa
$\nu = 0.3$
$H = 10$ GPa
$\sigma^o_{11} = \sigma^o_{22} = \sigma^o_{33} = 455$ MPa
material A: $\sigma^o_{12} = \sigma^o_{13} = \sigma^o_{23} = \sigma^o_{11}/\sqrt{3}$
material B: $\sigma^o_{12} = \sigma^o_{13} = \sigma^o_{23} = 0.5 \cdot \sigma^o_{11}/\sqrt{3}$
Load:
$p_{zo} = 10^{-3}$ MPa
$p_z = \lambda \cdot p_{zo}$

Figure 1: Geometrical and material data

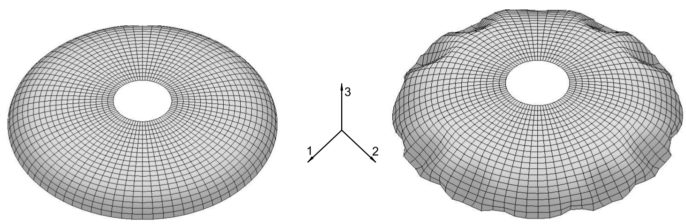


Figure 2: Deformed configuration for the materials A and B at displacement $w_C = 50$ mm

strains are concentrated at 45^o angle in $1-2$ plane.

Conclusion

An efficient numerical model for large strain elastoplastic material response has been presented in this paper . The model is based on the multiplicative decomposition of the deformation gradient and both kinematic and isotropic hardening are considered. The material employs orthotropic yield function expressed in terms of the Eshelby stresses and back stresses. The privileged directions of the material are defined by structural tensors. Unlike the spatial defined constitutive equations, mostly presented in the literature, the local integration algorithm and consistent tangent modulus are considered at the referent configuration. The example demonstrates the numerical stability and efficiency of the proposed algorithm. The computed deformed configuration presents the orthotropic structural behaviour, as expected.

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