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Fuzzy Optimization of Structures

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Summary

In this paper an extension of the crisp optimization of structures by considering uncertainty is presented. Uncertain parameters are described as fuzzy variables. This method is demonstrated by way of an example.

Introduction

The results obtained from classical methods of optimization involving deterministic variables exhibit various shortcomings. In particular, the effects of the uncertainty attached to all input information is often ignored altogether or only taken into account to a limited degree. The classical deterministic optimization problem according to Eq. (1) is considered under the aspect of uncertainty, and extended.

For the objective function $z(\underline{x}, \underline{e})$ the optimum solution $\underline{x}_{OPT}$ from the set of design variables $\underline{X}$ (design space) is determined under compliance with the equality constraints $h_j(\underline{x}, \underline{e})$ and the inequality constraints $g_i(\underline{x}, \underline{e})$. Input parameters such as geometrical parameters, material parameters, external load parameters, reliability parameters and economic parameters are lumped together in the vector $\underline{e}$.

$$\begin{aligned} &\text{find } \underline{x}_{OPT} \in \underline{X} \text{ with } z(\underline{x},\underline{e}) \Rightarrow \min \\ &\underline{X} = \left\{\underline{x} \,|\, g_i(\underline{x},\underline{e}),\ h_j(\underline{x},\underline{e})\right\} \quad g_i(\underline{x},\underline{e}) \le r_i \quad i = 1 \ldots n \\ &\qquad\qquad\qquad\qquad\qquad \text{and} \quad h_j(\underline{x},\underline{e}) = 0 \quad j = 1 \ldots m \end{aligned} \tag{1}$$

Uncertain parameters may be described using interval variables, stochastic variables (provided sufficient statistically-proven information is available) or, as selected in this case, by fuzzy variables. The uncertainty models may also be combined.

In this paper the input parameters $\underline{e} = \{e_1, e_2, ..., e_f\}$ are modeled as the fuzzy variable $\tilde{\underline{e}}$. The design variables $\underline{x} \in \underline{X}$ are considered to be crisp variables, as it is assumed that decision-makers require crisp design variables for the structure concerned [1].

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Fuzzy optimization problem

Uncertain input parameters $\tilde{\underline{e}}$ may be present in the objective function as well as in the equality and inequality constraints of Eq. (1).

Uncertain external loads or system characteristic values in the equality constraints, which represent the deterministic system analysis, lead to uncertain inequality constraints. An uncertain limit $\tilde{\underline{r}}_i$ may be directly specified for the right-hand side of the inequality constraint, e.g. in the form of uncertain reliability parameters. The uncertainty of the decision-maker with regard to objectives is taken into consideration by means of an uncertain formulation of the objective function.

Considering the uncertain parameters to be fuzzy variables, the deterministic optimization problem expressed by Eq. (1) is extended to a fuzzy optimization problem.

$$\begin{aligned} &\text{find } \underline{x}_{OPT} \in \underline{X} \text{ with } \tilde{z}(\underline{x},\tilde{\underline{e}}) \Rightarrow \min \\ \underline{X} = \{\underline{x} \mid \tilde{g}_i(\underline{x},\tilde{\underline{e}}),\ \tilde{h}_j(\underline{x},\tilde{\underline{e}})\}\ &\tilde{g}_i(\underline{x},\tilde{\underline{e}}) \mathrel{\tilde{\leq}} \tilde{r}_i \quad i = 1\ldots n \\ &\text{and } \tilde{h}_j(\underline{x},\tilde{\underline{e}}) = 0 \quad j = 1\ldots m \end{aligned} \tag{2}$$

Fuzzy variables are indicated by a tilde. The cartesian product of the uncertain variables $\tilde{\underline{e}}$ is a fuzzy set which exists in addition to the design space. This is referred to as the fuzzy input set $\tilde{E}$.

On the basis of the latter a solution algorithm is presented for the case of uncertainty only in the inequality constraints of Eq. (2). The vector $\underline{e}$ of the input parameters is subdivided into a crisp part $\underline{e}$ and a fuzzy part $\tilde{\underline{e}}$. Only the crisp parameters $\underline{e}$ are taken into consideration in the objective function.

Fuzzy inequality constraints

The fuzzy loads and fuzzy parameters $\tilde{\underline{e}}$ are taken into consideration in the left-hand side of the inequality constraint given by Eq. (2). These are defined by specifying the fundamental set **E** and the membership function $\mu(e)$ ([1] - [4] , Fig. 1).

$$\tilde{e} = \{(e;\ \mu(e)) \mid e \in \mathbf{E}\} \tag{3}$$

For the right-hand side $\tilde{r}_i$ of the uncertain "$\tilde{\leq}$"relationship a membership function is also defined, which remains constant at a value of unity up to the deterministic limit $r_{i,\alpha=1}$ (Fig. 1), and subsequently takes on the form of monotonically decreasing function. This so-called soft restriction permits a certain degree of exceedence of the apparently secure limit $r_{i,\alpha=1}$ (dependent on the defined function) up to the limit $r_{i,\alpha=0}$.

In this approach the permissible region is restricted in an uncertain manner.

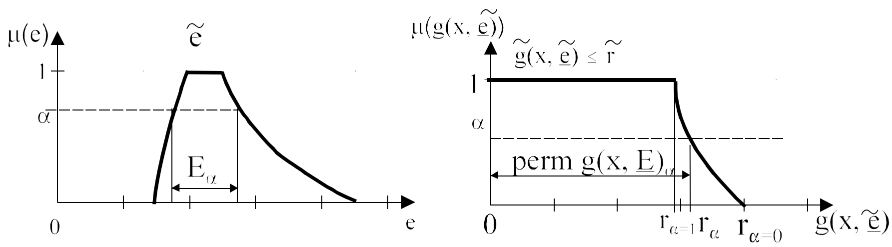

Fig. 1 Fuzzy variable, uncertain inequality constraint

Solution of the fuzzy optimization problem by means of α-discretization

The numerical solution of the fuzzy optimization problem is based on α-discretization [4]. On each α-level the fuzzy variable is described by the crisp α-level set E_α (see Fig. 1).

$$E_\alpha = \{e \in \mathbf{E} \,|\, \mu(e) \geq \alpha\},\ \alpha \in (0;\ 1] \tag{4}$$

The fuzzy optimization problem thus reduces to an inner and outer deterministic optimization problem on each α-level. The α-level sets E_α of all uncertain input parameters form the input subspace $\underline{E}_\alpha$.

$$\begin{aligned} &\alpha \in (0,1]:\ \text{find } \underline{x}_{OPT,\alpha} \in \underline{X}_\alpha \text{ with } z(\underline{x},\underline{e}) \Rightarrow \min \\ &\underline{X}_\alpha = \{\underline{x} \,|\, g_{i,\alpha}(\underline{x},\underline{E}_\alpha),\ h_{j,\alpha}(\underline{x},\underline{E}_\alpha)\}\quad g_{i,\alpha}(\underline{x},\underline{E}_\alpha) \leq r_{i,\alpha} \quad i = 1\ldots n \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{and}\quad h_{j,\alpha}(\underline{x},\underline{E}_\alpha) = 0 \quad j = 1\ldots m \end{aligned} \tag{5}$$

By comparing Eqs. (2) and (5) it follows that the effect of α-discretization is to subdivide the fuzzy inequality constraints $\tilde{g}_i(\underline{x},\underline{\tilde{e}}) \leq \tilde{r}_i$ into deterministic inequality constraints.

$$\tilde{g}_i(\underline{x},\ \underline{\tilde{e}}) \ \tilde{\leq}\ \tilde{r}_i \Leftrightarrow \alpha \in (0,\ 1]\ \forall\ \underline{e} \in \underline{E}_\alpha:\ (g_{i,\alpha}(\underline{x},\underline{e})) \leq r_{i,\alpha} \tag{6}$$

The fulfillment of $g_{i,\alpha}(\underline{x},\ \underline{e}) \leq r_{i,\alpha}$ requires that all elements $\underline{e} \in \underline{E}_\alpha$ satisfy Eq. (6).

The $\underline{e}_i^* \in \underline{E}_\alpha$, is sought which leads to $g_{i,\alpha}(\underline{x}^{[it]},\ \underline{e}_i^*) \Rightarrow \max$ for a fixed $\underline{x}^{[it]}$. The determination of $\underline{e}_i^*$ forms the inner optimization problem.

$$\text{find } \underline{e}_i^* \in \underline{E}_\alpha,\ \text{so that } g_i(\underline{x},\ \underline{e}_i^*) \rightarrow \max \wedge g_i(\underline{x},\underline{e}_i^*) \leq r_{i,\alpha} \tag{7}$$

This inner optimization problem must be solved for each inequality constraint. For the case of monotonic mapping of the uncertain input parameters $\underline{\tilde{e}}$ in the inequality constraints $g_{i,\alpha}(\underline{x}, \underline{e}_i) \leq r_{i,\alpha}$ a check of the corner points of the input subspace is sufficient.

After solving the inner optimization problems for all inequality constraints and $\underline{x}^{[it]}$ a new design variable for the outer optimization problem may be investigated.

The optimization algorithm may be summarized as follows:

(1) Choose a (new) design variable $x^{[it]}$ in the iteration step it.
(2) Find $\underline{e}_i$* for the inequality i using Eq. (7)
(3) Have all inequalities been dealt with? no ⇒ (2), Yes ⇒ (4)
(4) Are all inequalities fulfilled? no ⇒ (1), yes ⇒ (5)
(5) Compute the objective function value of $\underline{x}^{[it]}$
(6) Has the objective function value been improved? yes ⇒ $\underline{x}_{OPT} = \underline{x}^{[it]}$ ⇒ (7), no ⇒ (7)
(7) Is the termination criterion satisfied? yes ⇒ END, no ⇒ it =: it +1, (1)

In order to solve the inner and outer optimization problems modified evolution strategies are applied [4]. In [5] the solution strategy for the inner optimization problem is also referred to as so-called anti-optimization.

Due to fuzzy uncertainty in the right-hand side of the inequality constraint the permissible region increases as the α-level decreases (see Fig. 1). The solution with an improved objective function value may lie in the enlarged region. Uncertain input parameters, as in the following example, reduce the permissible region.

Example

This example demonstrates the effects of uncertain loading on the optimum result in a computation of the optimum cross-section of a framework made of aluminum with a density of 0.1 lb/in³ and an elasticity modulus of 10000 ksi. The load is modeled as a fuzzy parameter.

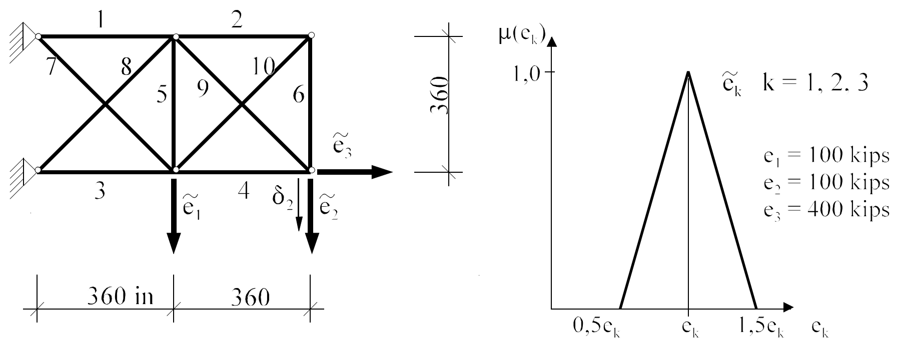

Fig. 2 System and loading

The right-hand sides of the inequality constraints (in this case permissible stresses perm σ in bars 1 to 10 and a permissible displacement perm δ_2) are modeled crisply, perm σ = 25 ksi and perm σ = 75 ksi, respectively for bar 9. The permissible displacement δ_2 is 5 in. The minimum cross-sectional area is 0.1 in².

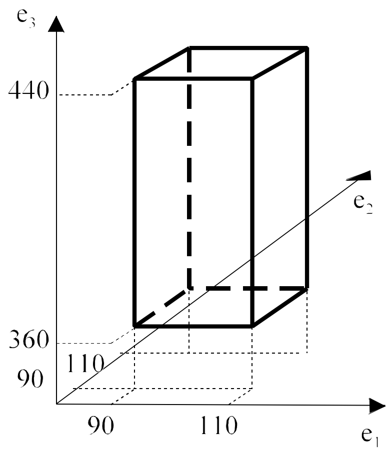

Fig. 3 Input subspace of the α-level: α = 0.8

A geometrically and physically linear computation is carried out. The results of the inner optimization problems for the level α = 0.8, $\underline{x}_{OPT,\alpha=0.8}$ are listed in Table 1.

Table 1 Optimum elements $\underline{e}_i$* of the input subspace α = 0.8 and $\underline{x}_{OPT,\alpha=0.8}$

bar / i	1	2	3	4	5
$\underline{e}_i$*	$\begin{pmatrix}110.0\\110.0\\360.0\end{pmatrix}$	$\begin{pmatrix}90.0\\110.0\\360.0\end{pmatrix}$	$\begin{pmatrix}90.0\\90.0\\440.0\end{pmatrix}$	$\begin{pmatrix}90.0\\90.0\\440.0\end{pmatrix}$	$\begin{pmatrix}90.0\\110.0\\440.0\end{pmatrix}$

6	7	8	9	10	δ_2 / 11
$\begin{pmatrix}90.0\\110.0\\360.0\end{pmatrix}$	$\begin{pmatrix}110.0\\110.0\\440.0\end{pmatrix}$	$\begin{pmatrix}110.0\\110.0\\360.0\end{pmatrix}$	$\begin{pmatrix}110.0\\110.0\\440.0\end{pmatrix}$	$\begin{pmatrix}90.0\\110.0\\360.0\end{pmatrix}$	$\begin{pmatrix}110.0\\110.0\\360.0\end{pmatrix}$

The results of the outer optimization problem for different load uncertainty and different α-levels are listed in Table 2. The larger cross-sectional areas necessary for an uncertain load lead to larger structural volumes. A more robust load-bearing behavior is thus guaranteed.

Whereas a statically determined framework is obtained as the optimum framework under crisp loading, a consideration of uncertainty leads to a statically undetermined framework as the optimum framework.

Table 2 Optimum cross-sectional areas [in²]

	$\alpha = 1.0$	$\alpha = 0.8$	$\alpha = 0.6$	$\alpha = 0.4$	$\alpha = 0.2$
bar 1	4.0353	4.4639	5.9136	7.9104	9.4929
bar 2	0.1	0.1	0.1	0.1	0.1
bar 3	4.0353	6.823	9.8867	12.855	15.607
bar 4	12.1	14.041	16.023	18.013	20.006
bar 5	3.8646	4.36063	4.1457	3.8895	3.8246
bar 6	0.1	0.1	0.1	0.1	0.1
bar 7	11.263	12.377	12.416	12.226	12.307
bar 8	0.1	0.1003	1.6219	3.4105	4.9408
bar 9	2.7577	4.6254	6.2954	7.3543	8.5169
bar 10	0.1414	0.1	0.1	0.1	0.1
weight	1598.6	1949.9	2342.4	2718.8	3085.6

Conclusions

By taking uncertainty into consideration it is possible to arrive at improved (more robust or lighter) designs with the aid of structural optimization. Future investigations will concentrate on the solution of the multi-criteria problem, uncertain objective functions and uncertain models.

Reference

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