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Coupled Electro-Stretching-Bending Analysis of Holes in the Electro-Elastic Composite Laminates

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Summary

Recently, by extending the Stroh formalism for two-dimensional linear anisotropic elasticity, we developed a Stroh-like formalism for the coupled stretching-bending analysis of composite laminates. Due to the resemblance of Stroh formalism and Stroh-like formalism, it has been observed through these two formalisms the solutions for the corresponding two-dimensional problems and coupled stretching-bending problems are really very alike. In addition, like the extension of Stroh formalism to the anisotropic piezoelectric materials, recently we also extended the Stroh-like formalism to the coupled analysis for the electro-elastic composite laminates. Therefore, it is expected that with the help of Stroh formalism, one may easily solve the problems of electro-elastic composite laminates by using the extended Stroh-like formalism. In this paper, the problems of the electro-elastic composite laminates containing elliptical holes are therefore solved analytically. Several numerical examples are performed for the multilayered composites made up of graphite/epoxy fiber-reinforced composite layers and PZT4 piezoelectric materials. The numerical results are well agree with those calculated by ANSYS finite element software package for the resultant forces, bending moments and electric displacements around the hole boundary.

Introduction

Due to the rapid development of intelligent space structures and mechanical systems, advanced structures with integrated self-monitoring and control capabilities are increasingly becoming important. It is well known that piezoelectric materials have the ability of converting energy from one form (among electric and mechanical energies) to the other. In other words, these materials can produce an electric field when deformed and undergo deformation when subjected to an electric field. If a multilayered composite is made up of different layers such as fiber-reinforced composite layers and composite layers consisting of the piezoelectric materials, it may exhibit electric effects that are more complicated than those of single-phase piezoelectric materials. According to the intrinsic coupling phenomenon, piezoelectric materials are widely used as sensors and actuators in intelligent advanced structure design. Recently, more advances are the smart or intelligent materials where piezoelectric materials are involved. Therefore, it is interesting to study the electro-elastic composite laminates.

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Since the analysis of the piezoelectric materials had been focused recently, there have been many research efforts on the electro-elastic materials analysis. However, because of the multiple coupling effects, it seems that the problems related to the electro-elastic composite laminates are not easy to be solved analytically, especially when the laminates contain defects such as cracks, holes or inclusions. In this paper through the analogy between the Stroh formalism [1,2] and our recent-developed extended Stroh-like formalism [3] we obtain the analytical solutions for an elliptical hole embedded in the unbounded electro-elastic composite laminates and use the numerical solutions obtained by ANSYS finite element software package to compare our results.

Mathematical Formulation

In a fixed rectangular coordinate system x_i, i=1, 2, 3, let σ_{ij}, ε_{ij}, D_i and E_i be, respectively, the stress, strain, electric displacement and electric field. The constitutive relation, equilibrium and electrostatic equations for the piezoelectric anisotropic elasticity are

$$\begin{aligned} &\sigma_{ij} = C^E_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \quad D_j = e_{jkl}\varepsilon_{kl} + \omega^S_{jk}E_k, \\ &\sigma_{ij,j} = 0, \quad D_{i,i} = 0 \quad i,j,k,l = 1,2,3, \end{aligned} \tag{1}$$

where repeated indices imply summation, a comma stands for differentiation, and C^E_{ijkl}, e_{kij} and ω^S_{jk} are the elastic stiffness tensor at constant electric field, piezoelectric stress tensor and dielectric permittivity tensor at constant strain, respectively, which are assumed to be fully symmetric. By letting

$$D_j = \sigma_{4j}, \quad E_k = -2\varepsilon_{4k}, \quad j,k = 1,2,3, \tag{2}$$

and

$$\begin{aligned} C_{IJKL} &= C^E_{IJKL}, \quad I,J,K,L = 1,2,3, \\ &= e_{LIJ}, \quad K = 4, \quad I,J,L = 1,2,3, \\ &= e_{JKL}, \quad I = 4, \quad J,K,L = 1,2,3, \\ &= -\omega^S_{JL}, \quad I = K = 4, \quad J,L = 1,2, \\ &= 0, \quad K = L = 4 \text{ and/or } I = J = 4, \end{aligned} \tag{3}$$

the constitutive relation $(1)_{1,2}$ can be rewritten in an *expanded tensor notation* as

$$\sigma_{IJ} = C_{IJKL}\varepsilon_{KL}, \quad I,J,K,L = 1,2,3,4. \tag{4}$$

In two-dimensional state, the relation (4) can be further reduced to [3]

$$\sigma_{pq} = C_{pqrs}\varepsilon_{rs}, \quad p,q,r,s = 1,2,4, \tag{5}$$

in which C_{pqrs} is for plane strain and short circuit condition. A replacement of equivalent elastic stiffness should be made for the other two-dimensional conditions, such as plane strain and open circuit condition, generalized plane stress and short circuit condition and generalized plane stress and open circuit condition. Detailed presentation of these equivalent stiffnesses can be found in [3].

Based upon the equilibrium and electrostatic equations shown in $(1)_{3,4}$ and the reduced constitutive relation (5) for two-dimensional problems, and the Kirchhoff's assumptions for the displacement and electric fields, by the way similar to the classical lamination theory [4] we may now write down the generalized constitutive relations and the generalized equilibrium equations for the electro-elastic composite laminates as [3]

$$\begin{aligned} &N_{pq}=A_{pqrs}u_{r,s}+B_{pqrs}\beta_{r,s}, \quad M_{pq}=B_{pqrs}u_{r,s}+D_{pqrs}\beta_{r,s}, \\ &N_{pj,j}=0, \quad M_{ij,ij}=0, \quad M_{4j,j}=0, \quad p,q,r,s=1,2,4;\ i,j=1,2. \end{aligned} \tag{6}$$

where N_{pj}, M_{pj} are the generalized resultant forces and bending moments; u_1 and u_2 are the mid-plane displacements in the x_1 and x_2 directions; β_1 and β_2 are the slopes related to the deflection w by $\beta_1=-w_{,1}$, $\beta_2=-w_{,2}$; u_4 and β_4 are the generalized mid-plane displacement and slope which are related to the electric fields $E_i=E_i^{(0)}+x_3E_i^{(1)}$, i=1,2, by $u_{4,i}=-E_i^{(0)}$, $\beta_{4,i}=-E_i^{(1)}$; A_{pqrs}, B_{pqrs} and D_{pqrs} are the tensor notations of the generalized extensional, coupling and bending stiffness, $A_{\alpha\beta}$, $B_{\alpha\beta}$, $D_{\alpha\beta}$, $\alpha,\beta=1,2,6,7,8$, which are related to the generalized elasticity stiffness $C_{\alpha\beta}$ [3].

Note that in equations $(1)_{3,4}$ and $(6)_{3,4,5}$, the body forces are neglected and the top and bottom surfaces of the laminates are free of traction and electric charge. For the cases that these values exist, the complete solutions should be found by adding a particular solution to the homogeneous solutions provided by the extended Stroh-like formalism shown below.

To solve the system of partial differential equations (6), the Stroh-like formalism developed by Hwu [5] for the coupled stretching-bending analysis of composite laminates has been used to treat the coupled electro-stretching-bending cases [3]. The general solutions satisfying all the basic equations shown in (6) have been written in Stroh-like form as

$$\mathbf{u}_d=2\,\mathrm{Re}\{\mathbf{Af}(z)\}, \quad \boldsymbol{\phi}_d=2\,\mathrm{Re}\{\mathbf{Bf}(z)\}, \tag{7}$$

where Re stands for the real part and

$$\mathbf{u}_d = \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\beta} \end{Bmatrix}, \quad \boldsymbol{\phi}_d = \begin{Bmatrix} \boldsymbol{\phi} \\ \boldsymbol{\psi} \end{Bmatrix}, \tag{8}$$

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ u_4 \end{Bmatrix}, \quad \boldsymbol{\beta} = \begin{Bmatrix} \beta_1 \\ \beta_2 \\ \beta_4 \end{Bmatrix}, \quad \boldsymbol{\phi} = \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_4 \end{Bmatrix}, \quad \boldsymbol{\psi} = \begin{Bmatrix} \psi_1 \\ \psi_2 \\ \psi_4 \end{Bmatrix}. \tag{9}$$

$$\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{a}_4 \ \mathbf{a}_5 \ \mathbf{a}_6], \quad \mathbf{B} = [\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3 \ \mathbf{b}_4 \ \mathbf{b}_5 \ \mathbf{b}_6], \tag{10}$$

In (9), $\phi_p, \psi_p, p = 1,2,4,$ are the stress functions related to the generalized resultant forces and moments by [3]

$$\begin{aligned} &N_{p1} = -\phi_{p,2}, \ N_{p2} = \phi_{p,1}, \ M_{p1} = -\psi_{p,2} - \frac{1}{2}\lambda_{p1}\psi_{k,k}, \ M_{p2} = \psi_{p,1} - \frac{1}{2}\lambda_{p2}\psi_{k,k}, \\ &Q_1 = -\frac{1}{2}\psi_{k,k2}, \ Q_2 = \frac{1}{2}\psi_{k,k1}, \ V_1 = -\psi_{2,22}, \ V_2 = \psi_{1,11}, \quad p = 1,2,4; \ k = 1,2. \end{aligned} \tag{11}$$

$\mathbf{f}(z)$ is a function vector composed of six holomorphic complex functions which will be determined through the satisfaction of the boundary conditions. The argument $z_k (= x_1 + \mu_k x_2), \ k = 1,2,...,6,$ of each component function $f_k(z_k)$ contains μ_k which is the material eigenvalue that has been proved to have six pairs of complex conjugates. $\mathbf{A}$ and $\mathbf{B}$ defined in (10) are two 6×6 complex matrices of which $\mathbf{a}_k$ and $\mathbf{b}_k$ are the material eigenvectors which can be determined by the material properties through the Stroh's eigen-relation [3].

Hole Problems

Consider an unbounded electro-elastic composite laminate with an elliptical hole subjected to the generalized forces and moments $N_{11}^{\infty}, N_{22}^{\infty}, N_{12}^{\infty}$, $M_{11}^{\infty}, M_{22}^{\infty}, M_{12}^{\infty}$ and $N_{41}^{\infty}, N_{42}^{\infty}, M_{41}^{\infty}, M_{42}^{\infty}$ at infinity. The contour of the elliptical hole is represented by

$$x_1 = a\cos\varphi, \ x_2 = b\sin\varphi, \tag{12}$$

where $2a$, $2b$ are the major and minor axes of the ellipse and φ is a real parameter. If the hole edge and the upper and lower surfaces of the laminate are free of traction and electric charge, the boundary conditions of this problem can be expressed in terms of the augmented stress function $\boldsymbol{\phi}_d$ as

$$\begin{aligned} &\boldsymbol{\phi}_d = \boldsymbol{\phi}_d^{\infty} = x_1\mathbf{m}_2^{\infty} - x_2\mathbf{m}_1^{\infty} \quad \text{at infinity}, \\ &\boldsymbol{\phi}_d = \mathbf{0} \quad \text{around the hole boundary}, \end{aligned} \tag{13}$$

where

$$\mathbf{m}_1^{\infty T} = \{N_{11}^{\infty} \; N_{12}^{\infty} \; N_{41}^{\infty} \; M_{11}^{\infty} \; M_{12}^{\infty} \; M_{41}^{\infty}\},$$
$$\mathbf{m}_2^{\infty T} = \{N_{12}^{\infty} \; N_{22}^{\infty} \; N_{42}^{\infty} \; M_{12}^{\infty} \; M_{22}^{\infty} \; M_{42}^{\infty}\}. \quad (14)$$

Due to the resemblance between the present formalism and that for the coupled stretching-bending problems [5], the analytical solution for the present problem can be found by referring to the corresponding solution shown in [6] for the holes in composite laminates, which is

$$\mathbf{u}_d = \mathbf{u}_d^{\infty} - \mathrm{Re}\{\mathbf{A} < \varsigma_k^{-1} > \mathbf{B}^{-1}(a\mathbf{m}_2^{\infty} - ib\mathbf{m}_1^{\infty})\},$$
$$\boldsymbol{\phi}_d = \boldsymbol{\phi}_d^{\infty} - \mathrm{Re}\{\mathbf{B} < \varsigma_k^{-1} > \mathbf{B}^{-1}(a\mathbf{m}_2^{\infty} - ib\mathbf{m}_1^{\infty})\}, \quad (15)$$

where $\mathbf{u}_d^{\infty}$ is the augmented displacement vector associated with $\boldsymbol{\phi}_d^{\infty}$ and

$$\varsigma_k = \frac{z_k + \sqrt{z_k^2 - a^2 - \mu_k^2 b^2}}{a - i\mu_k b}. \quad (16)$$

The real form solutions for the generalized resultant forces and moments around the hole boundary can also be obtained by referring to the corresponding solution shown in [6].

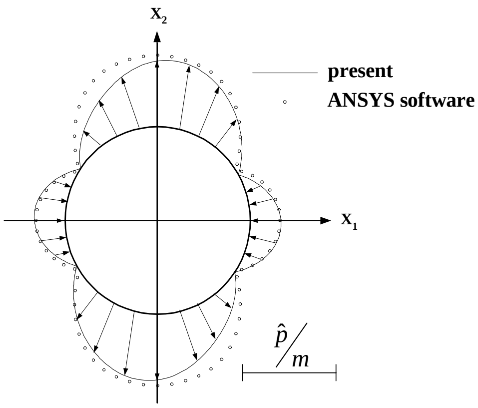

Figure 1. Hoop stress around the top surface of the circular hole boundary.

Figure 1 is a plot of the hoop stress around the top surface of the circular hole boundary when only in-plane force $N_{11}^{\infty} = \hat{p}$ is applied on the electro-elastic laminate [E/0/45/E] where E stands for PZT4 piezoelectric material and 0/45 are the fiber orientation of graphite/epoxy composite. The comparison is provided by ANSYS finite element software package using 3-D Solid 5 element with 72 nodes around the hole boundary and 1:15 hole/plate ratio to approximate the unbounded laminate. The stress contour diagram by ANSYS is shown in Figure 2.

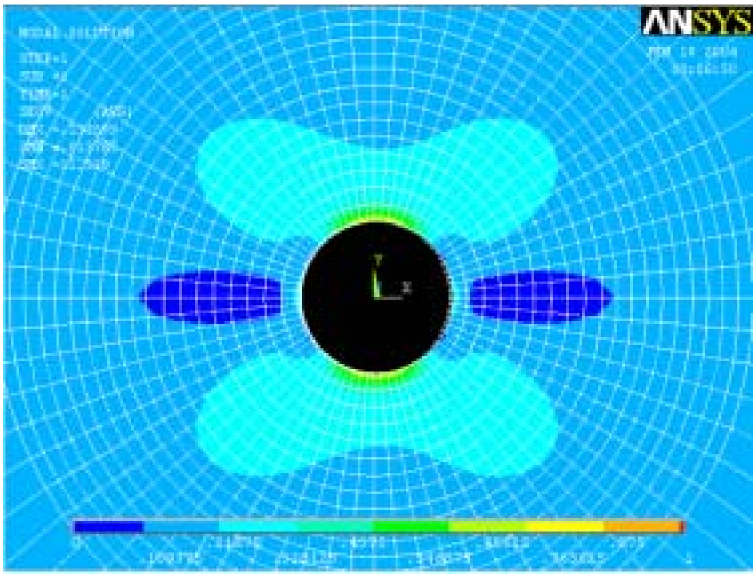

Figure 2. The stress contour by ANSYS 3-D Solid 5 element for an electro-elastic composite laminate with a circular hole (72 nodes around the hole boundary)

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