On the Continuity of Methods of Fundamental Solutions

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Summary

For the boundary value problems of homogeneous equations, we present a method to approximate the boundary value functions by using the fundamental solutions when the domain is a disc, which provides approximate solutions on the entire domain by the stability of the solutions of boundary value problems. Examples are given to show that our methods apply to Laplace, biharmonic, Helmholtz and modified Helmholtz equations.

1. Introduction

Consider the boundary value problem of a homogeneous equation in R^2

$$\mathcal{L}v(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Omega, \tag{1}$$
$$v(\mathbf{x}) = b(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega, \tag{2}$$

where $\mathcal{L}$ is a differential operator, Ω is a simply connected domain in R^2, and b is a boundary value function. Under appropriate conditions we assume that the above problem has a unique solution and it continuously depends on the boundary value function b, namely if b_n is a sequence of functions which converges to b in $L^\infty(\partial\Omega)$ and v_n is the sequence of the corresponding solutions for

$$\mathcal{L}v_n(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Omega, \tag{3}$$
$$v_n(\mathbf{x}) = b_n(\mathbf{x}), \qquad \mathbf{x} \in \partial\Omega, \tag{4}$$

then v_n converges to v in $L^\infty(\Omega)$ (cf. [1] for instance).

Denote by $G(\mathbf{x},\mathbf{y})$ the fundamental solution of (1), that is

$$\mathcal{L}G(\mathbf{x},\mathbf{y}) = -\delta(\mathbf{x}-\mathbf{y}), \qquad \mathbf{x},\mathbf{y} \in R^2, \tag{5}$$

where δ is the Dirac delta function. To solve (1)-(2) by the methods of fundamental solutions (MFS), we choose a fictitious domain $\widetilde{\Omega}$ such that the closure $\overline{\Omega} \subset \widetilde{\Omega}$, and choose source points $\mathbf{y}_k \in \partial\widetilde{\Omega}$, $1 \le k \le m$, and form a function

$$v_m(\mathbf{x}) = \sum_{k=1}^{m} c_k G(\mathbf{x},\mathbf{y}_k), \tag{6}$$

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where c_k, $1 \leq k \leq m$, are constants. Then v_m satisfies (1). For v_m to satisfy (2) as much as possible, collocation or least square methods are usually used (cf. [2] and references therein). For instance, in the collocation methods, one chooses m distinct points $\mathbf{x}_j \in \partial\Omega$ such that

$$v_m(\mathbf{x}_j) = h(\mathbf{x}_j), \qquad 1 \leq j \leq m, \tag{7}$$

which is then used to determine the coefficients c_k in (6) by solving the corresponding linear system. However the convergent results are difficult to obtain by the collocation methods. In [3,4], other approximation methods have been used to derive the convergent rates of MFS for the boundary value problems of Poisson's and modified Helmholtz equations. In the following Section 2 we will present the methods of approximating boundary value functions by using fundamental functions, and exmples will be explicitly shown in Section 3 to demonstrate that our methods apply to Laplace, biharmonic, Helmholtz and modified Helmholtz equations.

2. Approximation of Boundary Value Functions by Fundamental Solutions

For the sake of arguments, we identify R^2 with a complex plane, and assume that Ω and $\widetilde{\Omega}$ are concentric disks with the center at the origin, namely:

$$\Omega = \left\{re^{i\theta}; \theta \in [0, 2\pi]\right\}, \qquad \widetilde{\Omega} = \left\{Re^{i\theta}; \theta \in [0, 2\pi]\right\} \tag{8}$$

with $r < R$. For a given fundamental solution $G(\mathbf{x}, \mathbf{y})$, let

$$g(t) = G\left(re^{it}, R\right), \tag{9}$$

which is a 2π-periodic function of t. Expand g into its Fourier series

$$g(t) = \sum_{n=-\infty}^{\infty} c_n(g)\, e^{int}, \tag{10}$$

where $c_n(g) = \frac{1}{2\pi}\int_0^{2\pi} g(t)\, e^{-int} dt$ for any integer n. The following simple lemma is shown in [4].

Lemma 1. *Suppose that g is a 2π-periodic function and $c_n(g) \neq 0$ for any $n \in Z$. Then for any fixed integer m, if $-m \leq j \leq m-1$, there holds*

$$\left| e^{ijt} - \frac{1}{2mc_j(g)} \sum_{k=-m}^{m-1} e^{\frac{kji\pi}{m}} g\left(t - \frac{k\pi}{m}\right) \right| \leq \frac{1}{|c_j(g)|} \sum_{q\neq 0} |c_{j+2mq}(g)|. \tag{11}$$

Let g be given in (9) by a fundamental solution. For any $h(t) = \sum_{n\in Z} c_n(h) e^{int} \in L^\infty[-\pi,\pi]$, we define

$$\mathcal{A}_{m,k}h(t) := \sum_{\ell=-m}^{m-1} a_\ell(h,k)\, g\left(t - \frac{\pi\ell}{m}\right), \tag{12}$$

where k, m are integers, $k < m-1$, and

$$a_\ell(h,k) := \sum_{n=-k}^{k} c_n(h) \frac{e^{\frac{\ell n\pi i}{m}}}{2mc_n(g)}, \qquad -m \le \ell \le m-1. \tag{13}$$

It then follows that

$$\begin{aligned} \|h - \mathcal{A}_{m,k}h\|_\infty &\le \left\| h - \sum_{n=-k}^{k} c_n(h) e^{int} \right\|_\infty && (14)\\ &\quad + \left\| \sum_{n=-k}^{k} c_n(h) \left[e^{int} - \sum_{\ell=-m}^{m-1} \frac{e^{\frac{\ell n\pi i}{m}}}{2mc_n(g)} g\left(t - \frac{\pi\ell}{m}\right)\right]\right\|_\infty && (15)\\ &\le \sum_{|n|\ge k+1} |c_n(h)| + \sum_{n=-k}^{k} |c_n(h)| \sum_{q\ne 0} \frac{|c_{n+2mq}(g)|}{|c_n(g)|}, && (16) \end{aligned}$$

which we conclude in the following proposition.

Proposition 2. *Suppose that $g \in L^2[-\pi,\pi]$ satisfies $c_n(g) \ne 0$ for any $n \in Z$. Then for any $h \in L^\infty[-\pi,\pi]$, and any integers m, k, $k < m-1$, there holds*

$$\|h - \mathcal{A}_{m,k}h\|_\infty \le \sum_{|n|\ge k+1} |c_n(h)| + \sum_{n=-k}^{k} |c_n(h)| \sum_{q\ne 0} \frac{|c_{n+2mq}(g)|}{|c_n(g)|}. \tag{17}$$

To derive the convergent rates from the above proposition, the Fourier coefficients of h, g need to be estimated. If h is j times continuously differentiable, it is easy to see

$$|c_n(h)| \le c\frac{\|h^{(j)}\|_\infty}{|n|^j}, \qquad n \ne 0, \tag{18}$$

where c is a constant independent of h and for $n = 0$ we obviously have $|c_0(h)| \leq c \left\| h^{(j)} \right\|_\infty$. Hence the first term in the right-hand side of (17) can be estimated as follows

$$\sum_{|n| \geq k+1}^{\infty} |c_n(h)| \leq c \frac{\left\| h^{(j)} \right\|_\infty}{k^{j-1}}, \qquad k \neq 0. \tag{19}$$

To estimate the second term in the right-hand side of (17), the properties of fundamental solutions need to be used. It turns out that for many fundamental functions $c_n(g)$ decays exponentially or in the order of $(r/R)^n$, which guarantees the convergence of the second term in the right-hand side of (17). A detailed discussion of $c_n(g)$ for different fundamental functions will be given in the next section.

We now turn to the boundary value problem (1)-(2). Let $h(t) = b\left(re^{it}\right)$, $t \in [-\pi, \pi]$. And set

$$v_{m,k}(\mathbf{x}) := \sum_{\ell=-m}^{m-1} a_\ell(h,k)\, G\left(\mathbf{x}, Re^{i\frac{\pi\ell}{m}}\right), \tag{20}$$

then $v_{m,k}$ satisfies the equation (1), and on the boundary $\partial\Omega$,

$$\begin{aligned} v_{m,k}(re^{it}) &= \sum_{\ell=-m}^{m-1} a_\ell(h,k)\, G\left(re^{it}, Re^{i\frac{\pi\ell}{m}}\right) && (21) \\ &= \sum_{\ell=-m}^{m-1} a_\ell(h,k)\, g\left(t - \frac{\pi\ell}{m}\right) && (22) \\ &= \mathcal{A}_{m,k}h(t). && (23) \end{aligned}$$

Hence, if $\mathcal{A}_{m,k}h$ converges to h in $L^\infty[-\pi, \pi]$; that is, $v_{m,k}$ converges to the boundary value function b on $\partial\Omega$, by the assumption on the continuity of the solutions, $v_{m,k}$ converges to the unique solution v of (1)-(2) on Ω.

3. Examples of Differential Equations

The fundamental solutions of Helmholtz and modified Helmholtz equations involve different kinds of Bessel functions I_k, J_k, K_k, and even Hankel functions H_k. The properties and identities of these special functions can be found in [5] .

Example 1. For the Laplace equation $\Delta u = 0$, its fundamental solution is

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \ln \|\mathbf{x} - \mathbf{y}\|, \tag{24}$$

which we have

$$g(t) = -\frac{1}{2\pi}\ln\left\|re^{it} - R\right\| = \sum_n c_n(g)\, e^{int}, \tag{25}$$

with

$$c_n(g) = \begin{cases} -\frac{1}{2\pi}\ln R, & n = 0, \\ \frac{1}{4\pi|n|}\left(\frac{r}{R}\right)^{|n|}, & n \neq 0. \end{cases} \tag{26}$$

It can be shown (cf. [4]) that for $R \neq 1$, the estimation in Proposition 2 becomes

$$\|h - \mathcal{A}_{m,k}h\|_\infty \leq c\,\|h\|_\infty \left(\frac{1}{k^{j-1}} + \left(\frac{r}{R}\right)^{2(m-k)}\right). \tag{27}$$

Example 2. For a modified Helmholtz equation $\Delta u - \lambda^2 u = 0$, the fundamental solution

$$G(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi}K_0(\lambda\,\|\mathbf{x} - \mathbf{y}\|). \tag{28}$$

The Fourier coefficients of the corresponding g are given by

$$c_j(g) = \frac{1}{2\pi}K_j(\lambda R)\, I_j(\lambda r) \neq 0, \qquad j \in Z. \tag{29}$$

A similar result to (27) has been shown in [3].

Example 3. Consider a Helmholtz equation $\Delta u + \lambda^2 u = 0$, whose fundamental solution is

$$G(\mathbf{x}, \mathbf{y}) = -\frac{1}{4i}H_0(\lambda\,\|\mathbf{x} - \mathbf{y}\|), \tag{30}$$

which the Fourier coefficients of the corresponding g are given by

$$c_j(g) = \frac{1}{2\pi}H_j(\lambda R)\, J_j(\lambda r), \qquad j \in Z. \tag{31}$$

Denote by δ_k the smallest positive zero of J_k, then it is known that

$$\delta_0 (= 2.4048...) < \delta_1 < \cdots < \delta_n < \cdots, \tag{32}$$

and $J_n(x) > 0$ for $x \in (0, \delta_n)$. Hence if $\lambda r < \delta_0$, then $c_n(g) \neq 0$. Usine the properties of Hankel and Bessel functions (cf. [5]), a similar result of (27) can be derived.

Example 4. We now consider a biharmonic equation $\Delta^2 u = 0$, which the fundamental solution is

$$G(\mathbf{x}, \mathbf{y}) = \frac{1}{8\pi} \|\mathbf{x} - \mathbf{y}\|^2 \ln \|\mathbf{x} - \mathbf{y}\|, \qquad \mathbf{x}, \mathbf{y} \in R^2. \tag{33}$$

We have $c_{-n}(g) = c_n(g)$ for any n, and

$$c_0(g) = \left(r^2 + R^2\right) \ln R^2 + 2r^2, \tag{34}$$

$$c_1(g) = -\frac{1}{2}\frac{r^3}{R} - rR - rR\ln R^2, \tag{35}$$

$$c_n(g) = \left(\frac{r}{R}\right)^n \frac{1}{n(n-1)} - \left(\frac{r}{R}\right)^{n+2} \frac{1}{n(n+1)}, \qquad n \geq 2. \tag{36}$$

Hence $c_n(g) \neq 0$ for any $n \in Z$. A similar result as in (27) certainly can be expected.

Reference

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