A Methodology for Star-shape Transient Heat Source Reconstruction

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Summary

In this work we consider the numerical problem of reconstruction of an unknown characteristic transient thermal source inside a domain. The finite difference θ-scheme applied to the transient heat conduction equation leads to a model based on a sequence of modified Helmholtz equation solutions. For each modified Helmholtz equation the characteristic star-shape source function may be reconstructed uniquely from the Cauchy boundary data. Using representation formulae we establish a reciprocity functional mapping functions that are solutions of the modified Helmholtz equation to their integral in the unknown characteristic support.

Introduction

Transient Heat Problem

The direct transient heat source initial boundary value problem consists in finding $u(x,t)$ with $(x,t) \in \Omega \times [0,T), T > 0$ given a boundary input $g(x,t)$ with $(x,t) \in \Gamma \times [0,T)$, an initial input $u_0(x)$ with $x \in \Omega$ and a source distribution $f(x,t)$ with $(x,t) \in \Omega \times [0,T)$ that verifies the problem :

$$(P_{g,f}) \begin{cases} \frac{\partial u}{\partial t} - \Delta u = f, & \text{in } \Omega \times [0,T); \\ u = u_0, & \text{in } \Omega \times \{0\}; \\ u = g, & \text{on } \Gamma \times [0,T). \end{cases} \tag{1}$$

It is well known that this direct problem is well posed with unique solution for regular data.

The inverse source problem that we address consists in the recovery of the source $f(x,t) \in \Omega \times [0,T)$, knowing the initial data in Ω and the Cauchy data in the boundary Γ for $t \in [0,T)$. We consider that the unique information available is given by only one measurement, say, the Neumann boundary measurements

$$\partial_\nu u = g^\nu, \text{ on } \Gamma \times [0,T). \tag{2}$$

corresponding to $g = 0,$ on $\Gamma \times [0,T)$, where ν is the boundary domain exterior normal.

The θ-scheme and the modified Helmholtz model for the transient heat Problem

One type of sources that can be uniquely reconstructed from Neumann boundary measurements in a model based on Poison equation with the Laplace operator Δ, are star-

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shaped characteristic sources. This uniquenes resulted may be easily extended to a modified Helmholtz equation based model and we present now an algorithm for moving transient source reconstruction in the heat equation. Let the source be given by

$$f(x,t) = \chi_{\omega(t)}(x), \text{ in } \Omega \times [0,T) \tag{3}$$

where $\omega(t), t \in [0,T)$ is a representation of the star shape source boundary. For one - dimensional problems it is a set with two points. For two or tree-dimensional problems it is a moving Lipschitz parametric curve or surface in which the parameter has been omitted. With this in mind, we may rewrite the transient problem as

$$(P_g, \chi_\omega) \begin{cases} \frac{\partial u}{\partial t} - \Delta u = \chi_{\omega(t)}(x), & \text{in } \Omega \times [0,T); \\ -\Delta u_0 = \chi_{\omega(0)}(x), & \text{in } \Omega \times \{0\}; \\ u = g, & \text{on } \Gamma \times [0,T). \end{cases} \tag{4}$$

with transient Neumann history g^ν as in(2).
The initial u_0 can be determined as solution of the Poisson problem $-\Delta u_0 = \chi_{\omega(0)}$, if the initial shape $\omega(0)$ is known. If it is not known, then we can use the Cauchy data, $g(0)$ and $g^\nu(0)$, to solve the static inverse problem (cf. [1]). Consider a partition of the time interval $[0,T]$ into N subintervals of length $\tau > 0$. Let $\{t_0, t_1, t_2, ..., t_n, t_{n+1}, ...t_N\}$ be the knots of this partition, with $t_0 = 0$ and $t_N = T$. For $t_n < t < t_{n+1}, n = 0, 1, N-1$ we use the θ-scheme approach for the discretization of the (4). Define, for a function $h(x,t)$, a linear θ weighted approximation $\delta_\theta(h)(x)$ by

$$\delta_\theta(h)(x) = \theta h(x, t_{n+1}) + (1-\theta) h(x, t_n) \tag{5}$$

We start by approximating the time derivative $\frac{\partial u}{\partial t}$ in (4) by a first order forward difference

$$\frac{\partial u}{\partial t}(x,t) \cong \frac{u(x,t_{n+1}) - u(x,t_n)}{\tau}, \; x \in \Omega \tag{6}$$

the diffusion and the characteristic source respectively by

$$\Delta u(x,t) \cong \delta_\theta(\Delta u)(x), \; x \in \Omega \tag{7}$$

$$\chi_{\omega(t)}(x) \cong \delta_\theta(\chi_\omega)(x), \; x \in \Omega \tag{8}$$

By denoting u^{n+1} with $x \in \Omega$ the approximate solution at the time step t_{n+1} for this equation may be written as

$$(H^{n+1}_{g,\chi_\omega}) \begin{cases} -\Delta u^{n+1} + \lambda u^{n+1} = f_n + \chi_{\omega(t_{n+1})}, & \text{in } \Omega; \\ u^{n+1} = g(t_{n+1}), & \text{on } \Gamma; \\ \partial_\nu u^{n+1} = g^\nu(t_{n+1}), & \text{on } \Gamma. \end{cases} \tag{9}$$

for $n = 0, 1, 2, ..., N$. Here $\lambda = \frac{1}{\tau\ \theta}$ and $f_n = \frac{u^n + \tau(1-\theta)\Delta u^n + \tau\theta\chi_{\omega(t_n)}(x)}{\tau\theta}$. Note that $\Delta u^n + \chi_{\omega(t_n)} = \frac{\partial u^n}{\partial t}$ and for the initial framework

$$(H^0_{g,\chi_\omega}) \begin{cases} -\Delta u^0 = \chi_{\omega(0)}(x), & \text{in } \Omega; \\ u^0 = g(0), & \text{on } \Gamma; \\ \partial_\nu u^0 = g^\nu(0), & \text{on } \Gamma. \end{cases} \tag{10}$$

obtaining u_0 by solving the static inverse problem.

The sequence of modified Helmholtz source inverse problems (9) may be solved reconstructing the star-shape source $\chi_{\omega(t_n)}(x)$ for the time knots sequence, and shows its movement in the domain Ω.

Reciprocity functional

The reciprocity functional for the Helmholtz Problem depends only on boundary values of the solution and its properties are derived from elementary properties of the Green' theorem. Let v in be the space of Helmholtz functions $H^\lambda(\Omega) = \{v : -\Delta v + \lambda v = 0\}$. The reciprocity functional [1] for the Cauchy data in the sequence of Helmholtz problems (9) is

$$R^\lambda_{f_n+\chi_{\omega(t_{n+1})}}(v) = \int_\Gamma (vg^\nu(t_{n+1}) - g(t_{n+1})\partial_\nu v)ds, (\text{ for } v \in H_\lambda(\Omega)). \tag{11}$$

It is a direct consequence of Green's theorem that

$$R^\lambda_{f_n+\chi_{\omega(t_{n+1})}}(v) = \int_\Omega f_n v dx + \int_\Omega \chi_{\omega(t_{n+1})} v dx, (\text{ for } v \in H_\lambda(\Omega)). \tag{12}$$

The reciprocity functional can be calculated for different test functions $v \in H_\lambda(\Omega)$ using the Cauchy data of the respective modified Helmholtz problem or alternatively by the source function. Since at each time t_{n+1}, f_n may be calculated by using results from time t_n, with a procedure similar to that adopted in [1] to established a system of nonlinear equations for the source reconstruction inverse problem, we may form the sequence of moving sources reconstruction solutions for the system in (9).

One dimensional numerical simulations for the modified Helmholtz problem

The direct Helmholtz source problem consists in finding u given a boundary input $g \in H^{1/2}(\Gamma)$ and a source distribution χ_ω that verifies the Helmholtz-Dirichlet problem :

$$(H^\lambda_{g,\chi_\omega}) \begin{cases} -\Delta u + \lambda u = \chi_\omega, & \text{in } \Omega; \\ u = g, & \text{on } \Gamma. \end{cases} \tag{13}$$

It is well known that this direct problem is well posed with unique solution for $\lambda > 0$. The inverse source problem consists in by knowing the Cauchy data in the boundary Γ, that is

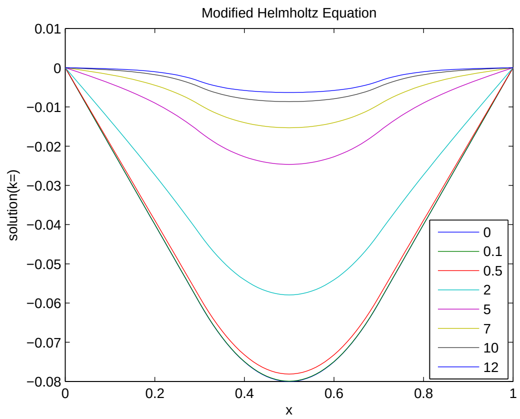

Figure 1: Slab modified Helmholtz equation solutions for $\sqrt{\lambda} = \kappa = 0, 0.1, 0.5, 2, 5, 7, 10, 12$.

the Dirichlet to Neumann map in at least one Dirichlet datum g, to recover the source χ_ω. This problem has been studied for generic sources by [4] who shown that it is useless to change the input Dirichlet data g. The unique information available is given by only one measurement, say, that Neumann boundary measurements

$$\partial_\nu u = g^\nu \tag{14}$$

corresponding to $g = 0$. The one dimensional model for the investigation of the λ parameter influence in the modified Helmholtz equation adopted is a slab with thickness equal $L = 1$ and a source supported in an interval centered in x_c and $h = b - a$ larger. The main advantages in uses the very simple simulations is that the greater majority of expressions may be deduced in an analytical way, by using elementary calculus. We consider Dirichlet boundary data $u(0) = u_0 = 0, u(1) = u_1 = 0$. The Neumann boundary are obtained with an semi-analytic solver. In figure 1 we shown a model case solution of the equation for $\sqrt{\lambda} = \kappa = 0, 0.1, 0.5, 2, 5, 7, 10, 12$.

Let $\{u_\lambda(0), u_\lambda(1)\}$ and $\{\frac{\partial u_\lambda(0)}{\partial x}, \frac{\partial u_\lambda(1)}{\partial x}\}$ be respectively the Dirichlet and Neumann data for the slab Helmholtz source problem with model (13). Then the reciprocity functional at the solutions $v_1^\lambda = \frac{sinh(\sqrt{\lambda}(x-x_c))}{\sqrt{\lambda}}$ and $v_2^\lambda = cosh(\sqrt{\lambda}(x-x_c))$, are, respectively,

$$R^\lambda_{v_1^\lambda}[x_c] = 0 = \frac{\partial u_\lambda(1)}{\partial x}\frac{sinh(\sqrt{\lambda}(1-x_c))}{\sqrt{\lambda}} - u_\lambda(1)cosh(\sqrt{\lambda}(1-x_c)) + \tag{15}$$

$$\frac{\partial u_\lambda(0)}{\partial x}\frac{sinh(\sqrt{\lambda}x_c)}{\sqrt{\lambda}} + u_\lambda(0)cosh(\sqrt{\lambda}x_c)$$

$$R^{\lambda}_{v_2^{\lambda}}[x_c, b-a] = \frac{2sinh(\sqrt{\lambda}(b-a)/2)}{\sqrt{\lambda}} = \frac{\partial u_\lambda(1)}{\partial x} cosh(\sqrt{\lambda}(1-x_c)) + \qquad (16)$$

$$u_\lambda(1)\sqrt{\lambda} sinh(\sqrt{\lambda}(1-x_c)) - \frac{\partial u_\lambda(0)}{\partial x} cosh(\sqrt{\lambda} x_c) + u_\lambda(0)\sqrt{\lambda} sinh(\sqrt{\lambda} x_c)$$

The centroid and source thickness determination is done by solving these equations. In figure 2 results for a slab with thickness $L = 1$, a interval characteristic source with centroid $x_c = 0.5$ and a source supported in the interval $h = b - a = 0.4$ in the modified Helmholtz equation for $\sqrt{\lambda} = \kappa = 0, 0.1, 0.5, 2, 5, 7, 10, 12$ are inserted in the Reciprocity functional at the hyperbolic sine and cosine function. The zeros of these functions gives the value of the centroid and interval of the characteristic one-dimensional source. It can be seen that the reciprocity functional intersects the horizontal axis in only on point, shown experientially that the determination of the source shape may be done by the reciprocity functional methodology.

Conclusions

We have presented a methodology for star shape source reconstruction in the transient heat problem by using one set of Cauchy data history. The method is based on a modified Helmholtz system based algorithm derived with the aid of finite differences time θ-scheme.

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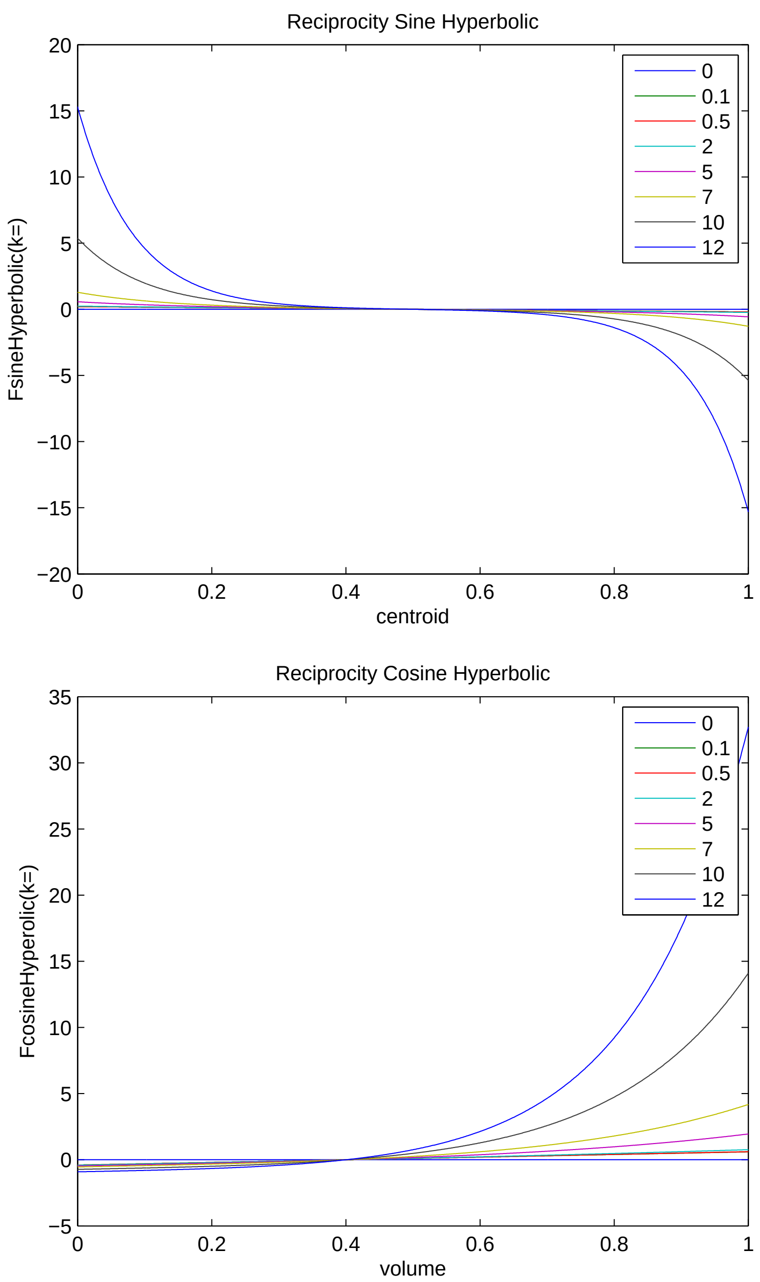

Figure 2: Slab modified Helmholtz equation solutions for $\sqrt{\lambda} = \kappa = 0, 0.1, 0.5, 2, 5, 7, 10, 12$ model . Reciprocity functional at a hyperbolic sine function for different centroid values. The same at a hyperbolic cosine function for centroid equal to 0.5 and different interval values.