Computation of periodic orbits for the Navier-Stokes equations

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We present a new method for computing periodic orbits of the Navier-Stokes equations that combines a finite-element discretisation in space with a spectral discretization in time. We illustrate our technique with calculations of the two-dimensional flow past a cylinder that is confined between the walls of a channel. Results for both rotating and non rotating cylinders will be presented.

When the Navier-Stokes equations are discretised in space using a finite-elment method (our technique would also work for a finite-difference or finite-volume method), we get a large system of ordinary differential equations

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x};\lambda); \quad \boldsymbol{f} : \mathrm{R}^N \times \mathrm{R} \mapsto \mathrm{R}^N. \tag{1}$$

where $\boldsymbol{x}$ is the vector of all the spatial degrees of freedom for the velocity and pressure. The velocity and pressure degrees of freedom are expanded in a Fourier series in time

$$\boldsymbol{x}(t) = \sum_{k=-K}^{K} \boldsymbol{x}_k \exp(i\omega t) \tag{2}$$

and the Galerkin method is applied to obtain a set of equations for the $\boldsymbol{x}_k$. A phase condition is added and this determines the period, $2\pi/\omega$, of the orbit. The method has two particularly attractive features: it preserves the circle group symmetry of the orbit; and it has an exponential rate of convergence as the number of Fourier modes is increased.

The steady laminar flow past the cylinder loses stability to a time-dependent flow at a supercritical Hopf bifurcation point. We compute the path of periodic orbits that emerge from this bifurcation point using our new method. When the cylinder does not rotate, the problem is symmetric about the mid-plane of the channel. Although the flows along the periodic orbit are not symmetric about channel mid-plane, it is still possible to exploit the symmetry to carry out the computation on the upper (or lower) half of the domain, thus reducing the cost significantly.

The periodic orbit that arises at the Hopf bifurcation point can disappear at a second Hopf bifurcation point at a higher value of the Reynolds number: the range of Reynolds numbers for which the orbit exists depends on the blockage ratio, i.e. the ratio of the channel width to the diameter of the cylinder. For sufficiently large blockage ratios (approximately greater than 0.85), there is no Hopf bifurcation from the symmetric steady flow. At these large values of the blockage ratio the flow first undergoes a steady symmetry-breaking bifurcation. There is then a pair of Hopf bifurcations, one on each of the asymmetric flows. The periodic orbits that emerge from these Hopf bifurcations have also been computed, but in this case it is no longer possible to exploit the reflectional symmetry to reduce the cost.

When the cylinder rotates, the reflectional symmetry of the problem is destroyed. If the cylinder rotation rate is high enough the flow is again stabilised and no Hopf bifurcation points are found for the steady flow, at least within the parameter ranges investigated here.

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