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How to describe turbulent energy distributions without the Fourier transform

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Abstract

It has been noted that the scale-by-scale distribution of kinetic energy in a turbulent flow is more readily observed in spectral space, using $E(k)$, than in real space, using the second-order structure function. For example, the 5/3rds law is usually easier to identify in experimental data than the equivalent 2/3rds law. We argue that this is not an implicit feature of a real-space description of turbulence. Rather, it is because the second-order structure function mixes small and large-scale information. In order to remedy this problem, Davidson[1] introduced a real-space function, $V(r)$, which plays the role of an energy density function, somewhat analogous to $E(k)$. In this paper we examine data taken in a variety of flows and determine the form of $V(r)$ in intermediate Reynolds number turbulence. We find that dissipation-range phenomena, such as the bottleneck effect, and the energy injection range are clearly evident in the signature function, but are absent in the structure function.

A Problem with the second-order Structure Function.

There are two common methods of describing how kinetic energy is distributed amongst the hierarchy of eddy sizes in isotropic turbulence. These are the three-dimensional energy spectrum, $E(k)$, and the second-order structure function, defined as,

$$\left\langle [\delta u(r)]^2 \right\rangle = \left\langle [u_x(\mathbf{x}+r\hat{\mathbf{e}}_x) - u_x(\mathbf{x})]^2 \right\rangle \quad .$$

The utility of the energy spectrum rests, in part, on three useful properties of $E(k)$:

1. it is positive;
2. it integrates to give the kinetic energy; and
3. a random distribution of eddies of fixed size l_e produces an energy spectrum of the form,

   $$E(k) \sim \left\langle \mathbf{u}^2 \right\rangle l_e (kl_e)^4 \exp\left[-(kl_e)^2/4\right] \quad ,$$

   which exhibits a peak at $k \sim \pi/l_e$ (see, for example, Ref.[1]).

The first two properties tells us that we may regard $E(k)$ as an energy distribution in spectral space, while the third suggests that we may loosely associate eddy size with π/k, at least in the range $\eta < \pi/k < l$, where l and η are the integral and Kolmogorov scales[2], respectively.

The second-order structure function can also be used to describe the distribution of energy over different scales. However, it has profound limitations, as we now show. The usual explanation for using $\left\langle [\delta u(r)]^2 \right\rangle$ as a measure of energy density is the following. Eddies of size much less than the separation r can induce a large signal at $\mathbf{x}$ or $\mathbf{x}' = \mathbf{x} + r\hat{\mathbf{e}}_x$, but not at both points simultaneously. Thus eddies smaller than r tend to induce a contribution to $[\delta u]^2$ which is of the order of their kinetic energy. On the other hand, eddies much greater than r tend to produce similar velocities at both $\mathbf{x}$ and $\mathbf{x}'$, and so make little contribution to the velocity difference, δu. So we might think of the structure function as a sort of filter, suppressing information from eddies of size greater than r. Given that $\frac{3}{4}\left\langle [\delta u]^2 \right\rangle \to \frac{1}{2}\left\langle \mathbf{u}^2 \right\rangle$ for large r we might expect that,

$$\begin{aligned} \frac{3}{4}\left\langle [\delta u(r)]^2 \right\rangle &\sim [\text{energy in eddies of size} < r] \\ &\sim \int_{\pi/r}^{\infty} E(k)dk, \end{aligned} \qquad (1)$$

and indeed such estimates are commonly made[3]. This led Townsend[3] to propose $V_T(r) = d\left\langle \frac{3}{4}[\delta u]^2 \right\rangle/dr$ (the subscript T to indicate Townsend) as a kind of energy density which plays a role analogous to $E(k)$. However, this is a deeply flawed view. Eddies whose sizes are much greater than r produce a contribution to $\frac{3}{4}[\delta u]^2$ of the order of $\frac{3}{4}\left\langle (\partial u_x/\partial x)^2 \right\rangle r^2 = \frac{1}{10}\left\langle \frac{1}{2}\omega^2 \right\rangle r^2$, so that we should replace Eq. (1) by the estimate,

$$\begin{aligned} \frac{3}{4}\left\langle [\delta u(r)]^2 \right\rangle \sim [\text{energy in eddies of size} < r] + \left(r^2/10\right) \\ \times [\text{enstrophy in eddies of size} > r]. \end{aligned} \qquad (2)$$

Indeed it is readily confirmed, using the transform pair which relates $E(k)$ to $\left\langle [\delta u(r)]^2 \right\rangle$, that a good approximation to the relationship between these two quantities is[1],

$$\frac{3}{4}\left\langle [\delta u(r)]^2 \right\rangle \approx \int_{\pi/r}^{\infty} E(k)dk + (r/\pi)^2 \int_0^{\pi/r} k^2 E(k)dk, \qquad (3)$$

which is precisely what we would expect from Eq. (2) [see Ref.[1] for a more detailed discussion of the progression from Eq. (1) to Eq. (3)]. Thus $\left\langle [\delta u(r)]^2 \right\rangle$ mixes large- and small-scale information, as well as information about energy and enstrophy. It follows that $V_T(r)$ is not a satisfactory estimate of the kinetic energy density. This failure of $V_T(r)$ led Davidson[1] to introduce a new function, called the *signature function*, which seeks to eliminate the large-scale information contained in Eqs. (2) and (3).

An Alternative to the Structure Function: The Signature Function.

The signature function is defined for isotropic turbulence only. It is:

$$V(r) = -\frac{1}{2} r^2 \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} \left\{ \frac{3}{4}\left\langle [\delta u(r)]^2 \right\rangle \right\}. \qquad (4)$$

It may be shown that $V(r)$ has the following properties[1]:

1. $$\int_0^r V(r)dr \geq 0;$$

2. $$\int_0^\infty V(r)dr = \frac{1}{2}\left\langle \mathbf{u}^2 \right\rangle;$$

3. a random distribution of eddies of fixed size l_e gives rise to the signature function,

$$V(r) \sim \left\langle \mathbf{u}^2 \right\rangle l_e^{-1} (r/l_e)^3 \exp\left(-r^2/l_e^2\right),$$

which has a peak around $r \sim l$.

If we compare these properties with those of $E(k)$ we see that, like the energy spectrum, $V(r)$ may be thought of as an energy density, with r interpreted as eddy size. The formal relationship between $E(k)$ and $V(r)$ is readily shown to take the form of a Hankel transform[1],

$$rV(r) = \frac{3\sqrt{\pi}}{2\sqrt{2}} \int_0^\infty E(k)(rk)^{1/2} J_{7/2}(rk)dk, \qquad (5)$$

from which it may be shown that,

$$rV(r) \approx [kE(k)]_{k=\hat{\pi}/r} \quad ,$$

where $\hat{\pi} = 9\pi/8$. For example, the difference between $rV(r)$ and $[kE(k)]_{k=\hat{\pi}/r}$ can be shown to be less than 4% for power-law spectra of the form[1],

$$E = Ak^n, \qquad -2 < n < 1.$$

One illustration of this is the 2/3rds law,

$$\left\langle [\delta u(r)]^2 \right\rangle = \beta \varepsilon^{2/3} r^{2/3}, \qquad (6)$$

(where β is the Kolmogorov constant and ε the mean turbulent energy dissipation rate) whose spectral equivalent is

$$E(k) = 0.761 \beta \varepsilon^{2/3} k^{-5/3}.$$

In terms of $V(r)$ we have, from Eq. (4),

$$rV(r) = \frac{1}{3}\beta \varepsilon^{2/3} r^{2/3} = 1.016\, [kE(k)]_{k=\hat{\pi}/r} \quad . \qquad (7)$$

The aim of the present work is to evaluate $V(r)$ for data taken from various flows. We will demonstrate that $V(r)$ is a superior diagnostic tool to $\left\langle [\delta u(r)]^2 \right\rangle$ for examining the energy structure of turbulence in the scale range $\eta < r < L$. Although this paper limits itself to "second-order" quantities, it is possible to define "higher-order" signature functions[1, 4]. For example, the natural progression from $V(r)$ is to define a signature function that, in some way, corresponds to the turbulent kinetic energy transfer flux and is comparable to the third-order structure function.

An Analytical Example

A simple example which demonstrates the utility of Eq. (4) is the following. Batchelor[5, 6] introduced a model expression for the second order structure function in the equilibrium range. It is exact in the limit $r \ll \eta$ and $r \gg \eta$, and interpolates between these limits. It is:

$$15\left\langle [\delta u(r/\eta)]^2 \right\rangle / u_\kappa^2 = \frac{(r/\eta)^2}{\left[1 + (15\beta)^{-3/2}(r/\eta)^2\right]^{2/3}} \quad , \qquad (8)$$

where $\eta \left(\equiv \nu^{3/4}\varepsilon^{-1/4}\right)$ and $u_\kappa \left(\equiv \nu^{1/4}\varepsilon^{1/4}\right)$ are the Kolmogorov length and velocity scales respectively. Figure 1 shows both Eq. (8) [with $\beta = 2.06$[10]] and 3 times Eq. (4) normalized by $\varepsilon^{2/3} r^{2/3}$. A viscous bottleneck effect[11, 12, 13] is clearly observed for the signature function in the cross-over region between the dissipative and inertial ranges. In the dissipative range, the signature function, unlike the structure function, quickly decays to zero as $r/\eta \to 0$. Both functions begin to display the required 2/3rds scaling from the same value of $r/\eta \approx 100$.

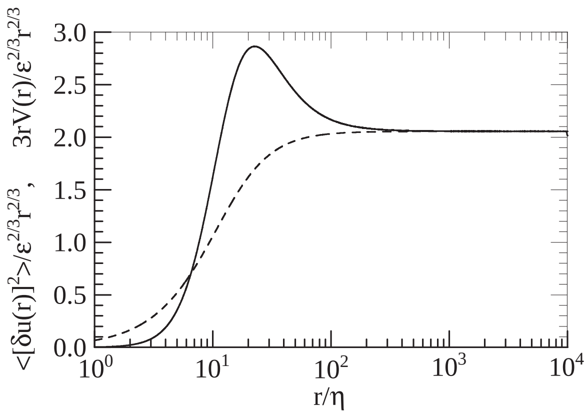

Figure 1: Curves of $3rV(r)$ and $\left\langle [\delta u(r)]^2 \right\rangle$, normalized by $\varepsilon^{2/3} r^{2/3}$ and plotted against r/η, for Batchelor's parametric form, Eq. (8). ———, $3rV(r)$; – – – –, $\left\langle [\delta u(r)]^2 \right\rangle$.

The Experimental Determination of the Signature Function.

The Experimental Data

The experimental data used for the present study are measurements made in fully developed turbulent wakes flows. Detailed experimental conditions can be found in Refs. [7, 8, 9] and need not be repeated here. The majority of data is acquired in a simple inexpensive geometry, which we call a NORMAN grid, that "stirs" vigorously on large scales. The geometry is composed of a perforated plate superimposed over a bi-plane grid of square rods. The grid is located in a blow-down wind-tunnel[7] of test section dimensions 35×35 cm^2 and 2 m length. The central three rows of the original bi-plane grid (mesh size $M = 50$ mm, original solidity $\sigma = 33\%$) have alternate meshes blocked (final $\sigma = 46\%$). As well as the NORMAN grid geometry, normal plate and circular cylinder wake data are re-evaluated here with original details also found in Ref.[7]. The measurements are made on the centreline of the wake formed behind each geometry at a downstream measurement station of $x/d \approx 40$. For all flows, signals of u are acquired on the mean shear profile centreline.

All data are acquired using the constant temperature anemometry (CTA) hot-wire technique with a single-wire probe made of $1.27\mu m$ diameter Wollaston (Pt-10% Rh) wire. The instantaneous bridge voltage is buck-and-gained and the amplified signals are low-pass filtered f_{lp} with the sampling frequency f_s always at least twice f_{lp}. The resulting signal is recorded with 12-bit resolution. Throughout this work, time differences τ and frequencies f are converted to streamwise distance r $(\equiv \tau U)$ and 1-dimensional longitudinal wavenumber k_1 $(\equiv 2\pi f/U)$ respectively using Taylor's hypothesis. The mean dissipation rate ε is estimated assuming isotropy of the velocity derivatives i.e. $\varepsilon \equiv \varepsilon_{\text{iso}} = 15\nu\langle(\partial u/\partial x)^2\rangle$. We estimate $\langle(\partial u/\partial x)^2\rangle$ from the average value of $E_{1D}(k_1)$ [the 1-dimensional energy spectrum of u such that $u^2 = \int_0^\infty E_{1D}(k_1)dk_1$] and from finite differences $\langle(\partial u/\partial x)^2\rangle = \langle u_{i+1} - u_i\rangle^2/(U f_s)^2$. For most of the data, the worst wire resolution is $\approx 2\eta$ where η is the dissipative length scale $\equiv \nu^{3/4}\varepsilon_{\text{iso}}^{-1/4}$. We have chosen not to correct for the decrease in wire resolution that is associated with an increase in R_λ, since all methods known to us rely on an assumed distribution for the 3-dimensional energy spectrum. For most of the data, the worst wire resolution is $\approx 2\eta$. For the NORMAN grid data, the worst wire resolution is $\approx 4\eta$.

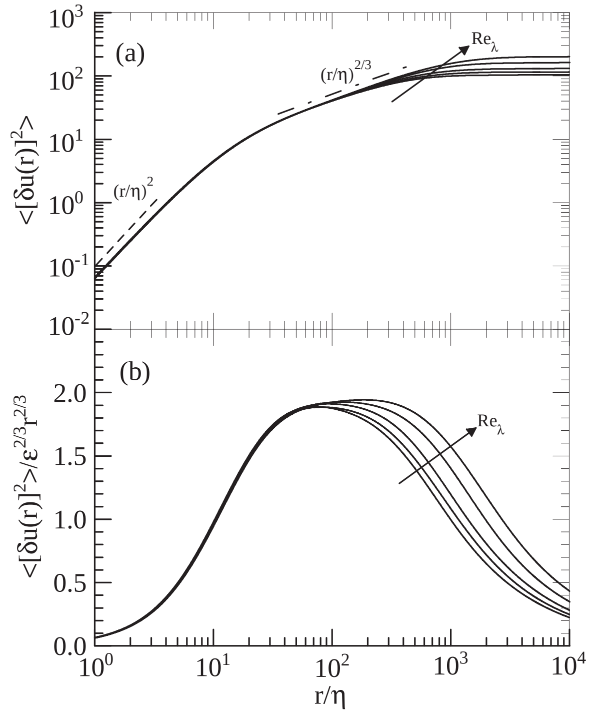

Figure 2: The Re_λ dependence of (a), $\left\langle[\delta u(r)]^2\right\rangle$; and (b), $\left\langle[\delta u(r)]^2\right\rangle$ normalized by $\varepsilon^{2/3}r^{2/3}$ and plotted against r/η, for NORMAN grid turbulence over the range $Re_\lambda = 200-400$. The $\nearrow$ arrow points in the direction of increasing Re_λ.

Results

Figures 2 and 3 show the Re_λ dependence of $\left\langle[\delta u(r)]^2\right\rangle$ and $3rV(r)$ respectively. The data are calculated from the measurements made in the NORMAN grid experiment[8, 9] over the range $Re_\lambda = 200-400$. Figure 2(a) shows that $\left\langle[\delta u(r)]^2\right\rangle \sim r^2$ as $r \to 0$ and $\left\langle[\delta u(r)]^2\right\rangle \sim r^{2/3}$ in the inertial range while Figure 3(a) shows that $3rV(r) \sim r^4$ as $r \to 0$ and $3rV(r) \sim r^{2/3}$ in the inertial range. The scaling behavior in the range $\eta \ll r \ll L$ can be more easily gleaned from Figs. 2(b) and 3(b). For both quantities, Re_λ dependent behaviors are observed. The signature function has a higher amplitude at large Re, and has acquired the characteristic double hump shape seen in energy spectra, e.g. see Fig. 5 in Ref. [14]. This second hump occurs in the region of $r \approx L$.

In Figure 4 we compare $3rV(r)$ and $\left\langle[\delta u(r)]^2\right\rangle$, both normalized by $\varepsilon^{2/3}r^{2/3}$, for NORMAN grid turbulence at $Re_\lambda = 255$. According to Eq. (7) the inertial range in such compensated plots should show up as a plateau with a numerical value equal to β. It is clear that, because of the modest value of Re_λ, only a limited inertial range is discernable in the signature or structure functions. Never-the-less, the expected overshoot in energy at the junction of the inertial and dissipation ranges shows up clearly in the signature function, though not in the structure function. The cause of this overshoot, which has become known as the bottleneck effect[11, 12, 13], are the viscous forces[1].

Figure 5 shows compensated plots of $rV(r)$ for NORMAN grid turbulence ($Re_\lambda = 255$) and for wakes behind a plate ($Re_\lambda = 248$) and a cylinder ($Re_\lambda = 254$). This time we use a linear plot and restrict ourselves to $r < L$, which corresponds to $r/\eta \sim 300$. The inertial range shows up more clearly in these plots, with a Kolmogorov constant of around $\beta \approx 2.0$ in good agreement with the consensus value[10]. More details of the experimental determination of $V(r)$ may be found in Ref. [4].

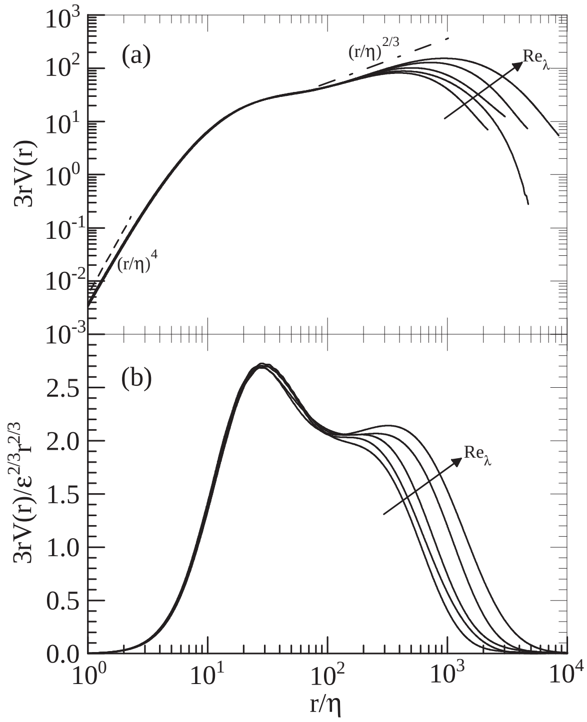

Figure 3: The Re_λ dependence of (a), $3rV(r)$; and (b), $3rV(r)$ normalized by $\varepsilon^{2/3}r^{2/3}$ and plotted against r/η, for NORMAN grid turbulence over the range $Re_\lambda = 200-400$. The $\nearrow$ arrow points in the direction of increasing Re_λ.

Final remarks and conclusions

We have explored the utility of the signature function $V(r)$ as an alternative to the structure function $\left\langle[\delta u]^2\right\rangle$ in describing turbulent kinetic energy in real space. At small separations r, the signature function $V(r)$, unlike $\left\langle[\delta u]^2\right\rangle$, captures the bottleneck behavior seen in Fourier space[11, 12, 13]. At large r, e.g. $r = L$, the signature function $V(r)$ indicates the region of large scale energy input. For the range of Re_λ investigated in the present work $V(r)$ clearly indicates that the large and small turbulent scales have insufficient separation for a "true" inertial range to exist.

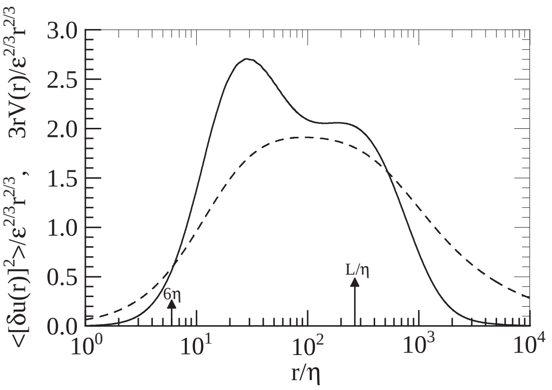

Figure 4: Curves of $3rV(r)$ and $\left\langle [\delta u(r)]^2 \right\rangle$, normalized by $\varepsilon^{2/3} r^{2/3}$ and plotted against r/η, for NORMAN grid turbulence, $Re_\lambda = 255$. ———, $3rV(r)$; – – – –, $\left\langle [\delta u(r)]^2 \right\rangle$. The first ↑ arrow indicates the average size of a dissipative structure and the second ↑ arrow indicates the integral length scale L.

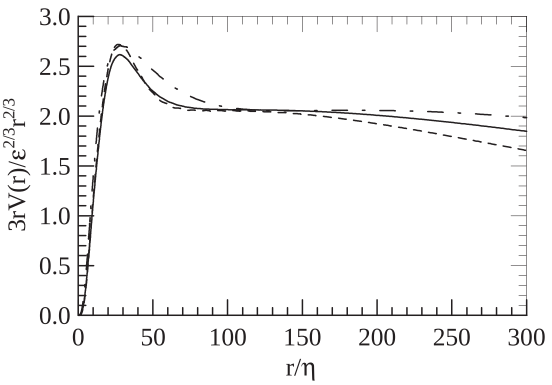

Figure 5: Compensated plots of $3rV(r)$ for: ———, plate wake turbulence $Re_\lambda = 248$; – – – –, Cylinder wake turbulence $Re_\lambda = 254$; and — – —, NORMAN grid turbulence $Re_\lambda = 255$.

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