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15th Australasian Fluid Mechanics Conference: On Collective Oscillations of Bubble Chains and Arrays

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Abstract

A number of different models have previously been developed to describe the collective oscillatory behaviour of gas-filled bubbles in a liquid medium. In this paper we perform an eigenanalysis on two mathematical models and discuss which is more physically realistic for the case of a chain of bubbles. The modal structures of both a bubble chain and bubble array are investigated, as well as the corresponding complex eigenfrequencies. For the case of two spherical bubbles located between two rigid parallel plates, we show how the eigenfrequencies change as the plate spacing is varied.

Introduction

Much work has been done towards the development of a model to describe the oscillations of gas bubbles in a liquid domain [1, 2, 3, 4, 5, 6], mostly analysing pairs of bubbles. The acoustically-coupled volumetric oscillations of bubbles are relevant in many fields, such as process engineering, ocean physics, microtechnology and medicine. Two different models for an arbitrary number of bubbles will be considered in this paper. The first model follows from work done by Manasseh *et al* [8], which has been developed for the particular case of a chain of bubbles. The second model, what we will call the standard model, is a simplified version of Feuillade's model [7], based on the theory of Tolstoy [2], and is applicable for any general configuration of bubbles. Both models appeared capable of predicting basic experimental trends [8, 9]. The models are discussed in the theory section of this paper. Numerical solutions for the eigenmodes and eigenfrequencies of a bubble chain are generated using each model. The results are compared and contrasted so that the most physically realistic model can be used.

We then look at an array of bubbles using the simplified standard model. A few modal structures are shown graphically, for the case of a square array of bubbles. The standard model is then modified to account for the presence of two rigid parallel plates between which two bubbles are trapped. The plates are modelled using the method of images. Using this very simplistic approach, we show that as the plates are brought together, the resonant frequencies (symmetric and asymmetric) decrease, which follows from work done by Strasberg [10]. The paper concludes with a summary of findings and outlines further work that is presently underway.

Theory

All the models used in this paper can be written in the following form:

$$\mathbf{M}\ddot{\mathbf{X}} + \mathbf{C}\dot{\mathbf{X}} + \mathbf{K}\mathbf{X} = \mathbf{0}, \tag{1}$$

where $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ represent inertial, damping, and stiffness matrices respectively and $\mathbf{X}$ is related to a differential bubble radius (i.e., the difference between the instantaneous and equilibrium bubble radii). Each model described below represents a system of second order differential equations with constant coefficients, the solution of which is harmonic in nature. Furthermore, equation 1 is a homogeneous equation (there is no driving term on the RHS) since we only require the natural frequencies and natural modes of a given bubble configuration. The coefficient matrices are determined by the assumptions made about the coupling between bubbles in the chain. For simplicity, it is assumed that all bubbles have equal radii.

Model 1

This is the model proposed by Manasseh *et al* [8]. To enable comparison with the standard model, equation 4 of [8] is reproduced here, and called Model 1A,

$$\ddot{\delta}_i + b_i\dot{\delta}_i + \omega_{0i}^2\delta_i = -\sum_{j\neq i}\frac{R_0}{s_{ji}}\left(\omega_{0j}^2\delta_j + b_j\dot{\delta}_j\right), \tag{2}$$

where δ is the change in bubble radius from an equilibrium radius R_0, $b = \omega_0^2 R_0/c$ is a radiative damping term, ω_0 is the radian frequency of a single, linearly oscillating spherical bubble in an unbounded liquid, s_{ji} denotes the spacing between centres of bubbles i and j, and c is the speed of sound in the liquid.

Model 1 was derived by assuming the coupling is due to the monopole superposition of the pressures from other bubbles. The bubbles can in principle have finite radii, whereas in the standard model, the bubbles are essentially point sources. Furthermore, there are no coupling terms arising from the velocity fields of neighbouring bubbles. In the course of Model 1's derivation, the liquid was first assumed to have a finite compressibility, and the compressibility was then made negligible. However the sign of the coupling term was negative because the bubbles were assumed from the outset to oscillate in a perfectly incompressible liquid. This appears to be an inconsistency, but since Model 1A had predicted experimental data, it was not clear if Model 1A was inappropriate. A self-consistent version is Model 1B,

$$\ddot{\delta}_i + b_i\dot{\delta}_i + \omega_{0i}^2\delta_i = \sum_{j\neq i}\frac{R_0}{s_{ji}}\left(\omega_{0j}^2\delta_j + b_j\dot{\delta}_j\right), \tag{3}$$

and in this paper, both 1A and 1B will be analysed to judge which is more appropriate.

Model 2

The model developed by Feuillade [7] was simplified by assuming that the acoustic wavelengths are much larger than the spacing between bubbles (effectively also neglecting liquid compressibility). Equation 7 in [7] has also been arranged to have the same form as equations 2 and 3 above, yielding:

$$\ddot{\delta}_i + b_i\dot{\delta}_i + \omega_{0i}^2\delta_i = -\sum_{j\neq i}\frac{R_0}{s_{ji}}\left(\ddot{\delta}_j\right). \tag{4}$$

Feuillade couples each bubble by scattered pressure fields from all other bubbles and then relates the pressure to the inertial mass of each bubble. This effectively couples the bubbles by velocity fields assumed continuous throughout a multiphase domain. Inherent in the model is that the bubbles act as radiating point sources. The pressure due to damping stress from other bubbles is not included. As an aside, this model is identical to that of Hsiao *et al* [5] when damping is neglected.

Numerical Method

A computer program was developed to find the eigenvalues (eigenfrequencies) and eigenvectors (eigenmodes) for a system of differential equations of the form given by equation 1. To do so, the system of equations was converted into state-space coordinates of the form:

$$\dot{\mathbf{Z}} = \mathbf{A}\mathbf{Z}, \tag{5}$$

where

$$\mathbf{Z} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix}, \quad \mathbf{Y} = \dot{\mathbf{X}}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}. \tag{6}$$

The program makes use of the numerical routines in *Numerical Recipes in C* as well as *CLAPACK* routines and libraries to calculate the eigenvalues and eigenvectors of equation 5. The required output was coded in C so as to produce a MATLAB-readable file (i.e., an M-file). The M-file was run in MATLAB to produce the plots shown in this paper.

The program reads an input file containing parameters of interest (e.g. bubble size, bubble separation, number of bubbles in the chain, etc.), creates the coefficient matrices from these values and according to the desired model, constructs the state-space matrix, then calculates the eigenvalues and eigenvectors.

Chains of Bubbles

A number of interesting plots were generated to show the similarities and differences between the models using a chain of bubbles. Firstly, the program was tested for the case of an undamped two-bubble chain to see if it reproduced the analytic natural frequencies. Once the numerical output was verified, the chain was extended to thirty bubbles, and plots of the modal structures were generated.

Two-bubble Chain

For Model 1A the analytic low-frequency eigenmode is given by, $\omega_1 = \sqrt{(1-R_0/s)}\omega_0$, while the high-frequency mode is given by, $\omega_2 = \sqrt{(1+R_0/s)}\omega_0$. For the standard model, $\omega_1 = \omega_0/\sqrt{(1+R_0/s)}$ and $\omega_2 = \omega_0/\sqrt{(1-R_0/s)}$. Figures 1 and 2 show the agreement between the analytical expressions and the numerical values generated by the program, as a function of the ratio of bubble separation to bubble radius, for each model.
As expected, figures 1 and 2 have the same general behaviour. The most important point is that the models break down when the bubbles are brought close together (the two natural frequencies rapidly diverge). In reality the bubbles would coalesce to form a single, larger bubble. (For a physical intuition behind the frequency shift of the two modes see [7].)

Eigenmodes for a Chain of Thiry Bubbles

Figures 3 and 4 show the first five consecutive modes with every fifth mode shown thereafter, for a chain of thirty bubbles. A bubble radius of 2.605 mm was used in the computation and the bubble separation was calculated such that all thirty bubbles fit within a chain of length 0.535 m. The centreline on each

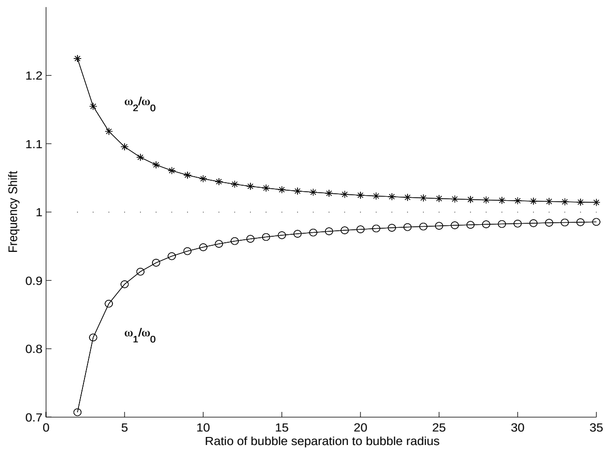

Figure 1: Frequency shifts of the natural frequencies for a two-bubble chain using Model 1A. The horizontal axis represents a non-dimensional ratio scaled in terms of bubble radii. The vertical axis scales the modal resonance frequency relative to the resonance frequency of a single bubble in free space. The solid lines represent the analytic solutions. The points denoted "$\circ$" and "$*$" are the results from the numerical eigenvalue solver.

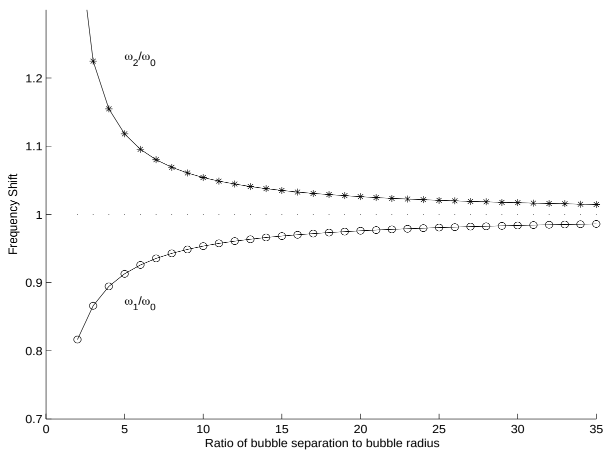

Figure 2: Frequency shifts of the natural frequencies for a two-bubble chain using Model 2. The same notation as shown in figure 1 is used.

plot represents the condition where bubble radii are at equilibrium. Also note that damping has been neglected since it has very little effect on the modal structures. Here we see a fundamental difference between models. For Model 1A, the lowest frequency mode corresponds to the most complicated mode structure, whilst the highest frequency mode corresponds to the simplest eigenmode (in which all bubbles oscillate in phase). This is contrary to physical intuition. For the standard model, however, the opposite is true; the lowest frequency mode corresponds to the simplest eigenmode, whilst the highest frequency mode corresponds to the most complicated mode. This is true of most harmonic situations (e.g. harmonics of a plucked string). When Model 1B is used, the eigenmode structure has a form consistent with physical intuition, like that of Model 2.

Also worth noting is that the bubbles towards the ends of the chain oscillate at a lower amplitude than those in the middle. This is analogous to a system of masses connected by springs, in which the masses in the middle have more flexibility than those on the end.

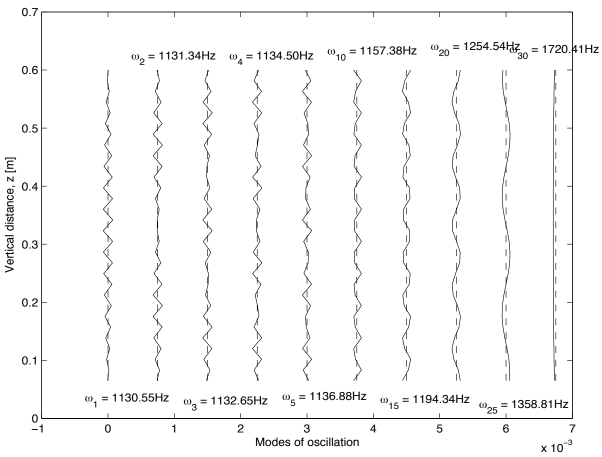

Figure 3: Modal structures for Model 1A for a chain of thirty bubbles. A chain height of 0.535 m was used. The lowest frequency mode has the most complicated mode structure. The highest frequency mode occurs when all bubbles oscillate in phase.

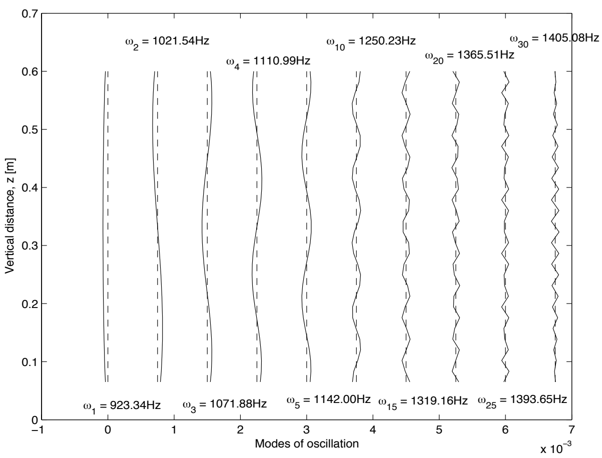

Figure 4: Modal structures for Model 2 for a chain of thiry bubbles. A chain height of 0.535 m was used. The lowest frequency mode has the simplest mode structure and corresponds to all bubbles oscillating in phase.

For each of Models, 1A, 1B and 2, it can be shown that the low-frequency mode is less damped than the high-frequency mode. This agrees with physical intuition that low frequency oscillations generally survive longer than higher frequency oscillations; the model of Hsiao *et al* [5] predicts the opposite. An interim conclusion is that Model 2 is both physically realistic and self-consistent and should be used in preference, although more comparison with experiment is warranted.

Arrays of Bubbles between Parallel Plates

This section of work is based on the standard model approach to the coupling between spherical bubbles. This model was used because it can be easily extended to any configuration of bubbles. However, it does mean that the bubbles are modelled as point sources, and so the model is invalid when the bubbles become too close.

To simulate the presence of top and bottom plates, the method of images was used. The configuration for the case of two bubbles is shown in figure 5. The plates are simulated by the presence of bubble images and therefore act as mirrors. Each bubble image is in phase with its respective original (i.e., B1's top and bottom images are in phase with itself). Note that only primary images have been considered. In fact, for a source between parallel plates the streamfunction is made up of an infinite series of images. Neglecting the other images is justified since they are further away and thus have less effect on the mass loading of the surrounding fluid.

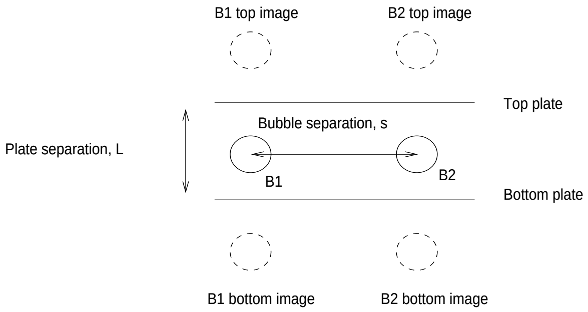

Figure 5: Configuration of the bubble image model for two bubbles, B1 and B2.

To generate meaningful results, the first task was to modify the numerical code developed earlier (for the bubble chain) for the case of a bubble array with no images. This involved respecifying the spacing between each and every bubble so as to produce an array rather than a chain configuration. An example of some of the eigenmodes that can be generated from such a configuration is shown in figure 6. This plot represents the situation in which the plates are infinitely far apart.

The standard model was then modified to include the presence of the bubble images. This meant adding extra inertia terms (due to the bubble images) to each bubble between the plates. The result was a set of N-coupled equations describing the oscillation of an N-bubble array between plates. The numerical code was adapted to take this into account by altering the inertia matrix.

The effect of the plate separation on the resonant frequencies is best shown for the case of two bubbles. This is shown in figure 7. As the ratio of plate separation to bubble radius (L/R_0) decreases, the two resonant frequencies decrease. This is supported by work done by Strasberg [10], in which he looks at the effect of a rigid boundary on the pulsation frequency of a spherical bubble. An intuitive way of explaining the above result is that the presence of the bubble images oscillating in phase with the original bubbles increases the mass loading, retards the motion and therefore reduces the resonant frequencies [7].

Conclusions

We have performed an eigenanalysis to determine the eigenfrequencies and eigenmodes of spherical bubbles in both chain and array configurations. The models assume that the bubbles are of identical radii and oscillate linearly, remaining spherical in form. For small ratios of bubble separation to bubble radius, the models clearly break down.

In comparing the two models, it has been noted that Model 2, the standard model, assumes that the bubbles act as point sources, whereas this assumption is not necessary for Model 1. From a physical point of view, both models agree with the intuition that lower frequency oscillations survive longer than higher frequency ones. However, considering the eigenmode structures for a chain of bubbles, Model 2 agrees with what is generally seen in nature (the lowest frequency mode has the simplest structure and the highest frequency mode the most

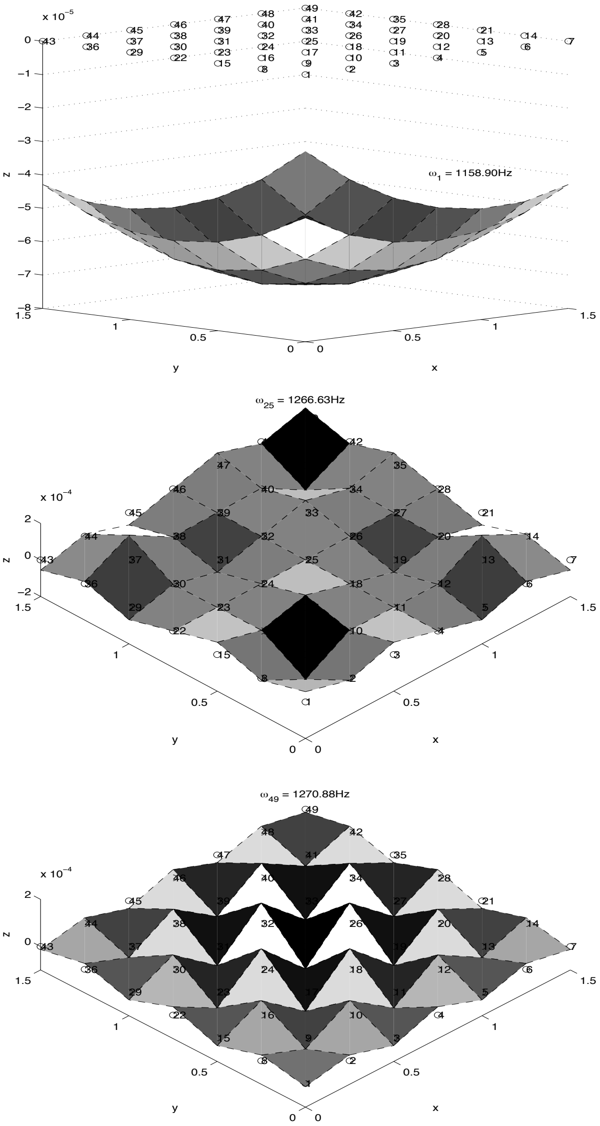

Figure 6: Eigenmode structures for the 1st, 25th and 49th modes of a 7×7 array of 49 bubbles. This was generated for identical bubble radii of 2.605 mm and without damping. The spacing between bubble 1 and bubble 2 is 0.25 m.

complex), whereas Model 1A predicts the opposite. A corrected version of Model 1, Model 1B, removes this problem. An interim conclusion is that the 'standard' Model 2 should still be used.

For the case of two bubbles bounded by parallel plates, we have shown that the resonant frequencies decrease as the plates are brought closer together. However, the analysis becomes inappropriate for small plate spacings, where the bubbles would no longer be spherical. On this note, current work in progress is aimed at investigating the resonant frequencies of bubbles which are trapped between parallel plates, not just bounded by them. Thus the bubbles are shaped more like cylinders than spheres. This work will hopefully determine what happens to the resonant frequencies of such bubbles as the plate spacing is varied. To some extent, the modified standard model for the case of two bubbles bounded by parallel plates offers some guidance for this work.

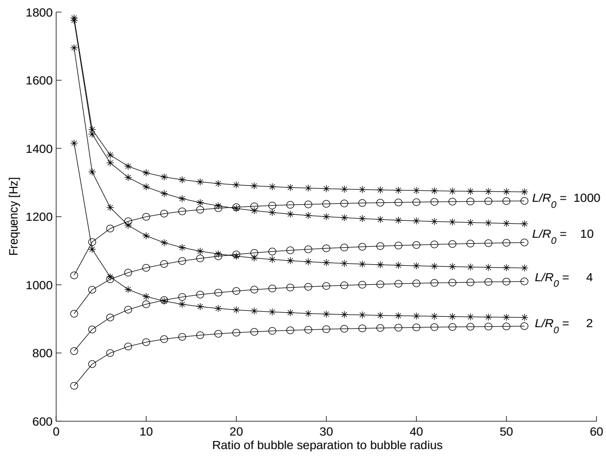

Figure 7: The resonant frequency curves for the two modes as plates with dimensionless spacing L/R_0 are brought closer together. This was generated for identical bubble radii of 2.605 mm and without damping.

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References

[1] Zabolotskaya, E. A., Interaction of gas bubbles in a sound field, *Sov. Phys. Acoust.*, **30**(5), 1984, 365–368.

[2] Tolstoy, I., Superresonant systems of scatterers. I., *J. Acoust. Soc. Am.*, **80**(1), 1986, 282–294.

[3] Ogũz, H. and Prosperetti, A., A generalization of the impulse and virial theorems with an application to bubble oscillations, *J. Fluid Mech.*, **218**, 1990, 143–162.

[4] Doinikov, A. A. and Zavtrak, S. T., On the mutual interaction of two gas bubbles in a sound field, *Phys. Fluids*, **7**(8), 1995, 1923–1930.

[5] Hsiao, P.Y., Devaud, M. and Bacri, J.C., Acoustic coupling between two air bubbles in water, *Eur. Phys. J. E*, **4**, 2001, 5–10.

[6] Ida, M., A characteristic frequency of two mutually interacting gas bubbles in an acoustic field. *Physics Letters A*, **297**(3–4), 2002, 210–217.

[7] Feuillade, C., Scattering from collective modes of air bubbles in water and the physical mechanism of superresonances, *J. Acoust. Soc. Am.*, **98**, 1995, 1178–1190.

[8] Manasseh, R., Nikolovska, A., Ooi, A. and Yoshida, S., Anisotropy in the sound field generated by a bubble chain, Journal of Sound and Vibration, 2004, *J. Sound Vibration*, **278**, 2004, 807–823.

[9] Nikolovska, A., Ooi, A., Widjaja, R., and Manasseh, R., Resonant scattering from a chain of air bubbles in water, *Seventh European Conference on Underwater Acoustics, ECUA 2004, Delft, The Netherlands*, 5–8 July 2004.

[10] Strasberg, M., The Pulsation Frequency of Nonspherical Gas Bubbles in Liquids, *J. Acoust. Soc. Am.*, **25**, 1953, 536–537.