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Three-Dimensional Quasi-Periodic Instabilities of Two-Dimensional Time-Periodic Flows

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Abstract

Three-dimensional synchronous instabilities of two-dimensional time-periodic wake flows, such as the so-called modes A and B of the circular cylinder wake, are now well-known. On the other hand, quasi-periodic three-dimensional instabilities are just as generic, but have as yet not received such wide-spread recognition, partly as a consequence of the predominance of attention given to flows, such as the cylinder wake, in which the synchronous modes happen to dominate at onset of three-dimensionality. Here we provide an introduction to the quasi-periodic modes, which can manifest either as modulated standing waves, or modulated travelling waves.

Introduction

Three-dimensional instabilities of flows with an underlying two-dimensional time-periodic state became a topic for investigation following experimental studies of secondary instabilities of the circular cylinder wake (figure 1), see e.g. [7]. The following analytical works, e.g. [1, 6] concentrated on Floquet stability analysis of wake flows, particularly those of the circular and square cylinders. The initial investigations dealt with the synchronous three-dimensional modes, i.e. those for which the critical Floquet multipliers pass through the unit circle at $\mu = +1$, along the positive real axis in the complex plane. For these wake flows, there were two synchronous modes: long-wavelength mode A, and short-wavelength mode B, as illustrated in figure 2. Modes A and B have different symmetry properties — mode A preserves, while mode B breaks, the spatio-temporal symmetry of the base flow — but these properties are the same across the two flows. The onset Reynolds number for mode A ($Re_c = 188$) is lower than that of mode B ($Re_c = 259$), but as Reynolds numbers increase, mode B becomes dominant, and a mixed-mode model for this behaviour has appeared [2].

The Floquet analysis of Barkley & Henderson [1] suggested the presence of another intermediate-wavelength Floquet mode for the circular cylinder wake, but with complex-conjugate-pair multipliers, and which would possibly bifurcate from the two-dimensional basic state at Reynolds numbers above those for either mode A or mode B. Addressing the wake of the square cylinder, Robichaux et al. [6] also suggested the existence of an intermediate-wavelength mode, but this time subharmonic, i.e. a mode for which the critical Floquet multiplier is $\mu = -1$. Later analysis [3] showed that for the square cylinder, this mode in fact also had complex-conjugate-pair multipliers, therefore the three-dimensional bifurcation scenarios for the wakes of the circular and square cylinder wakes are the same.

In general, we might expect that critical Floquet multipliers could emerge anywhere around the unit circle, i.e. at $\mu = +1$, $\mu = -1$, or $\mu = \exp \pm i\theta$. The wake flows have only a single control parameter (Reynolds number), and it happens that

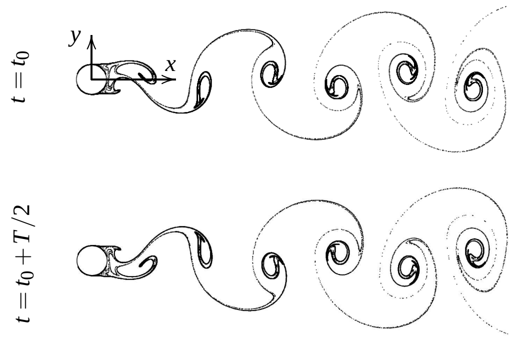

Figure 1: Computed locations of marker particles, illustrating the spatio-temporal symmetry of a two-dimensional circular cylinder wake for $Re = 188.5$ at times t_0 and $t_0 + T/2$ (t_0 is arbitrary, T is the Strouhal period).

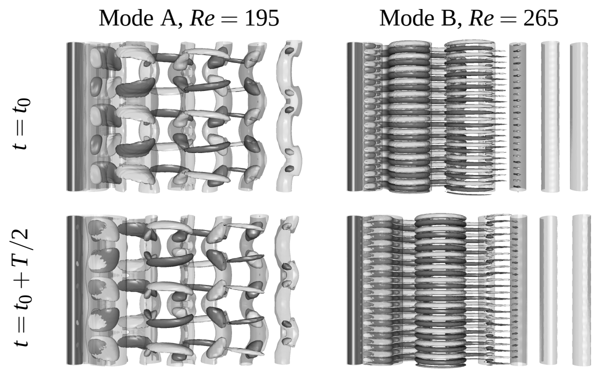

Figure 2: Vorticity isosurfaces for the synchronous wake modes of the circular cylinder, shown for a $10D$ spanwise domain extent, and viewed from the cross-flow direction. Translucent isosurfaces are for spanwise vorticity component, solid surfaces are for streamwise component.

the first modes to bifurcate from the two-dimensional basic state are synchronous. However, many other flows share the same symmetry group as the wake flows, and hence will have instability modes with the same symmetries. The flow in a rectangular cavity, infinite in spanwise extent, driven by a wall in periodic tangential motion (as illustrated in figure 3) is such an example, and further, possesses two independent control parameters; Reynolds and Stokes numbers. The Floquet analysis for this flow was presented in [4], where it was shown that within certain control-parameter regimes, either a

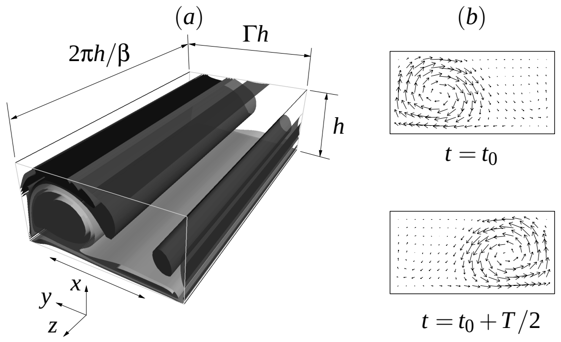

Figure 3: (*a*) Schematic of the fluid domain for the periodically driven cavity flow, periodic in the *z*-direction and forced in the *y*-direction, with isosurfaces representing different values of spanwise vorticity. (*b*) Two snapshots of velocity vectors of the base flow, half a forcing period apart, illustrating its spatio-temporal symmetry.

long-wavelength mode A, a short-wavelength mode B, or an intermediate-wavelength mode QP would be the first to become unstable. Interestingly, the symmetry properties of the synchronous modes (A and B) for this non-autonomous flow are the opposite to those of the same relative wavelengths for the circular and square cylinder wakes.

More recently, the complete analysis for all codimension-1 (generic) bifurcations for these flows was presented [5]. An important result of that work is that, if the base flows have Z_2 spatio-temporal symmetry (a time-shift of $T/2$, combined with a spatial reflection, regenerates the original flow, as in figures 1 and 3), then period-doubling bifurcations, while not suppressed, become codimension-2, i.e. can only be produced by careful simultaneous manipulation of two control parameters, and are unlikely to be observed when only one parameter is varied.

Another outcome was a more complete exposition of the nature of quasi-periodic modes. In general, there is no reason to expect that the synchronous modes will be dominant in all flows or parameter regimes, and it is just as likely that flows will arise in which the quasi-periodic modes are dominant. In the remainder of this paper, we give examples of these instability modes, and describe their properties.

Symmetries of the Quasi-Periodic Modes

When the Floquet multipliers occur in complex-conjugate pairs, then a new secondary period arises in the three-dimensional state, and the solution is quasi-periodic. In the cases under discussion, this period is associated with spanwise motion of the mode. In the physical domain, there are two ways that the quasi-periodic modes can manifest themselves, either as travelling waves (TW), which are unsymmetric but bifurcate in reflection-symmetric pairs, or as standing waves (SW), which have reflection symmetry but can be centred at arbitrary fixed spanwise locations. It is important to realise that in the nonlinear case (in DNS), the SW and TW evolve on distinct solution branches, with separate stability characteristics. The possible stability scenarios at a quasi-periodic bifurcation [5] are illustrated in figure 4.

Since the basic states have a spanwise translation symmetry, we use Fourier expansions in that direction to describe three-dimensional instabilities. The spanwise coordinate is z, the planar coordinates x and y, and the velocity $\mathbf{u} = (u, v, w)(x, y, z, t)$. Then

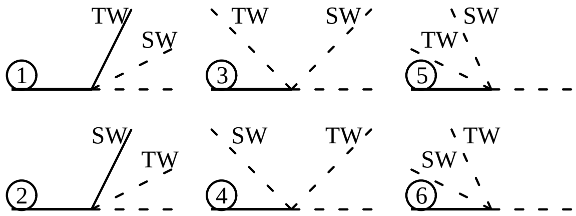

Figure 4: Bifurcation diagrams corresponding to the six scenarios in the QP bifurcation. Solid (dashed) lines represent stable (unstable) states, the horizontal line corresponds to the T-periodic base state. The horizontal axis is the bifurcation parameter (*Re*), and the vertical axis is the amplitude squared of the 3D components of the solution.

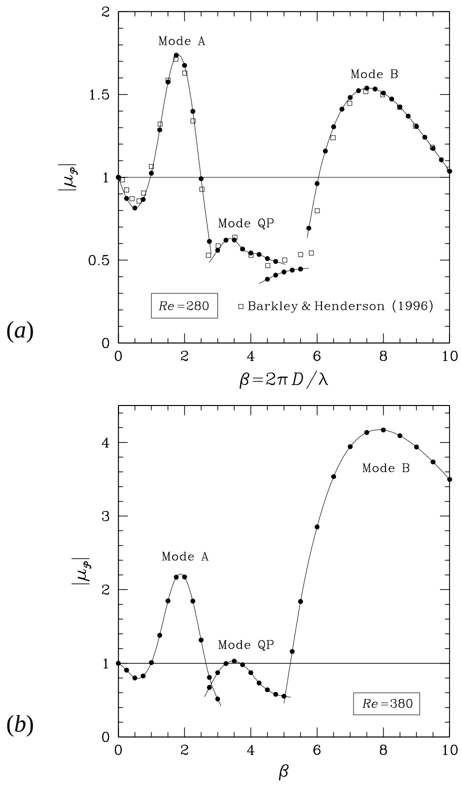

Figure 5: Moduli of the Floquet multipliers, $|\mu_{\mathcal{P}}|$, for the three-dimensional instability modes of the two-dimensional wake of a circular cylinder at (*a*) $Re = 280$ and (*b*) $Re = 380$. The $Re = 280$ results are compared with those of [1] at the same *Re* (open squares). Multipliers for mode QP occur in complex-conjugate pairs.

the following forms (which have a reflection symmetry in z) pass unchanged through the Navier–Stokes equations:

$$\mathbf{u}(x, y, z, t) = (u\cos\beta z, v\cos\beta z, w\sin\beta z)(x, y, t) \qquad (1)$$
$$\mathbf{u}(x, y, z, t) = (u\sin\beta z, v\sin\beta z, w\cos\beta z)(x, y, t), \qquad (2)$$

where β is a spanwise wavenumber: $\beta = 2\pi D/\lambda$, where λ is a spanwise wavelength and D a characteristic length scale. If the Floquet multipliers are real, then it is sufficient to use one of either of these forms, or a fixed linear combination of the two, to describe one of the infinite set of possible solutions. On the other hand, if the multipliers occur in complex-conjugate pairs, then restriction to any of these forms implies that the solutions

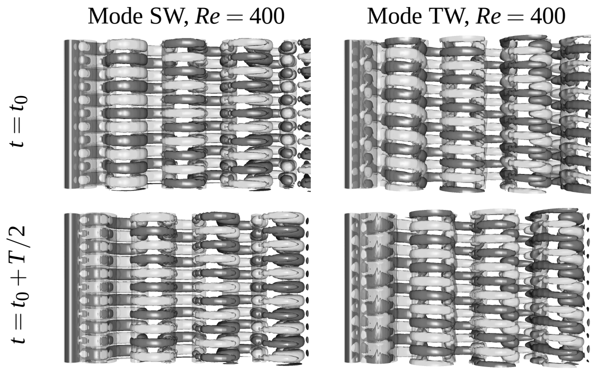

Figure 6: Vorticity isosurfaces for the quasi-periodic wake modes of the circular cylinder, shown for a 10D spanwise domain extent, and viewed from the cross-flow direction. Translucent isosurfaces are for spanwise vorticity component, solid surfaces are for streamwise component.

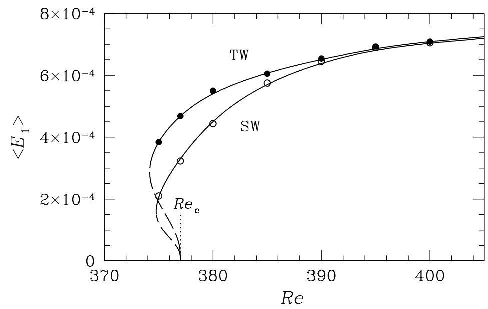

Figure 7: Time-average kinetic energies in the first spanwise Fourier mode of the TW and SW nonlinear solutions of the wake of a circular cylinder, as functions of Reynolds number. Solid (open) circles correspond to stable (unstable) solutions, relative to each other. Dashed lines indicate the unstable segments of the two solution branches, taken individually.

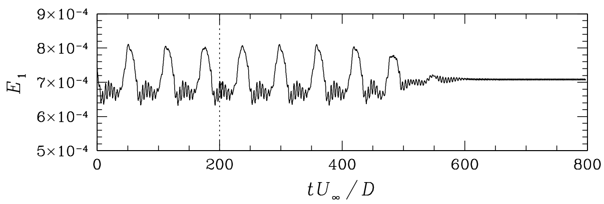

Figure 8: Time series of kinetic energy in the first spanwise Fourier mode for a quasi-periodic circular cylinder wake. Initially, the flow is in a SW state, in a subspace with spanwise reflection symmetry. At $tU_\infty/D = 200$, it is perturbed with a small amount of white noise, after which it evolves to a stable, asymmetric, TW state.

are SWs (which can remain in that state unless perturbed), and more generally we must consider non-symmetric expansion sets to allow for TW-type solutions [3, 4].

Quasi-Periodic Modes of the Circular Cylinder Wake

In figure 5 we show results from Floquet analysis for the wake of a circular cylinder, both at $Re = 280$, recreating results in [1], and at $Re = 380$. At $Re = 280$, the intermediate-wavelength mode QP is subcritical, while at $Re = 380$, it has just bifurcated from the basic state. At bifurcation, the complex-conjugate-pair Floquet multipliers lie quite close to the negative real axis.

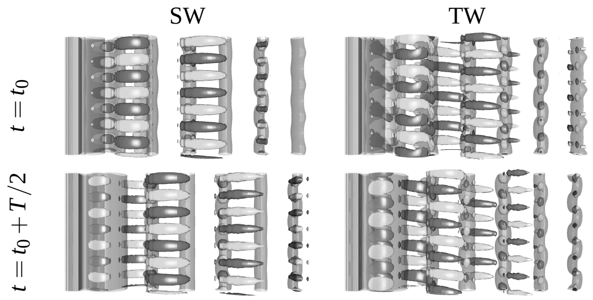

Figure 9: Vorticity isosurfaces for the quasi-periodic wake modes of the square cylinder at $Re = 220$, shown for a 10D spanwise domain extent, and viewed from the cross-flow direction. Translucent isosurfaces are for spanwise vorticity component, solid surfaces are for streamwise component.

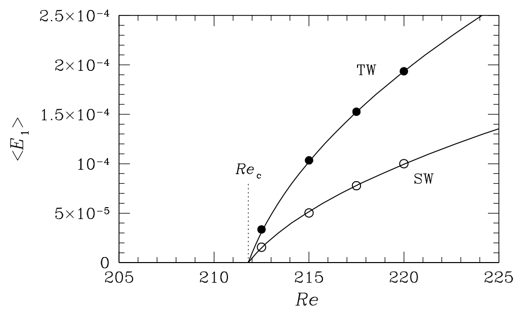

Figure 10: Time-average kinetic energies in the first spanwise Fourier mode of the TW and SW instability modes of the wake of a square cylinder, as functions of Reynolds number. The energy of the SW is smaller, and in accordance with the theory, the SW flow is unstable to perturbations.

In figure 6, isosurfaces of vorticity from $Re = 400$ DNS studies of the SW and TW circular cylinder wake modes are displayed. The TW shown is the downwards-travelling case. The visual distinguishing characteristic of the TW is that the streamwise vortices on opposite sides of the wake interlace one another by $\lambda/4$, while in all other cases (A, B, SW) they are in-line. Apart from that, in this flow they bear a number of visual similarities to mode B, and probably arise from a similar physical mechanism.

Growth and Relative Stability of the SW and TW Solutions

To highlight the point that the TW and SW solution branches are distinct under nonlinear evolution, we show in figure 7 the time-averaged energies in the first spanwise mode of the SW and TW solutions as functions of the bifurcation parameter, Re. Both bifurcations are subcritical, and the energy of the TW solutions is always larger: we have bifurcation scenario 6 in figure 4.

Typically, solutions on at most one of the two branches (SW, TW) are stable *relative to* those on the other branch. This means that if we slightly perturb a solution on the unstable branch, it will evolve to a solution on the other branch. The theory suggests that solutions on the lower-energy branch are those which are relatively unstable. This is exactly what is observed if a solution on the SW branch is slightly perturbed, as seen in figure 8, derived from a perturbation study at $Re = 400$. At $tU_\infty/D = 0$, the solution is a SW, with long-period oscillations in the energy of the first spanwise mode (the long period is closely related to the imaginary part of the corresponding Floquet multiplier). At $tU_\infty/D = 200$, a small $[O(10^{-4})]$ perturba-

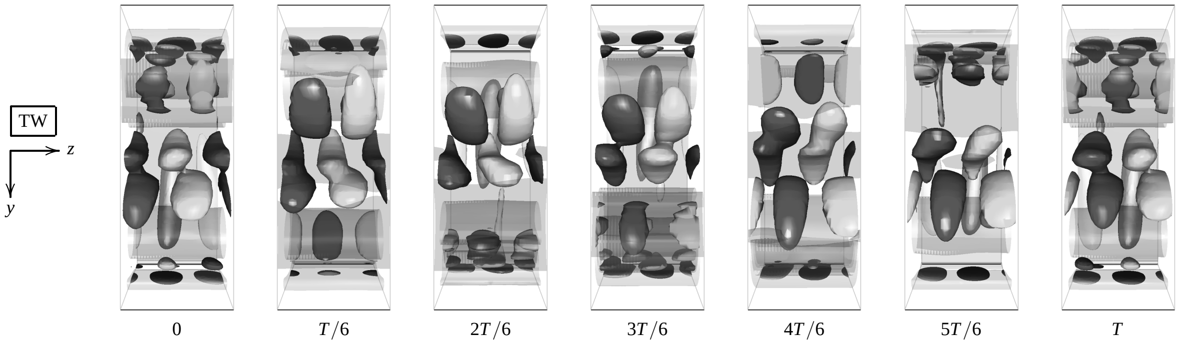

Figure 11: Vorticity dynamics of modulated $+z$-traveling waves in the periodically driven cavity flow [4], shown in a spanwise (z) domain extent of one wavelength, at ($St = 100$, $Re = 1225$). Solid isosurfaces are of the out-of-page (x) component of vorticity, positive and negative of equal magnitude, while translucent isosurfaces represent the z component of vorticity. The driven cavity wall lies further into the page than the structures, and oscillates in the $\pm y$ direction.

tion is administered to the real and imaginary parts of the first spanwise mode. For a further $\Delta t U_\infty/D \approx 200$, there is little observable change in the energy, but by $tU_\infty/D = 500$ the system has settled down to a stable TW state, and the long-period fluctuation in energy disappears. The TW is a three-dimensional solution that translates in the spanwise direction with constant kinetic energy (modulo T).

Quasi-Periodic Modes of the Square Cylinder Wake

The quasi-periodic bifurcation of the wake of a square cylinder provides very similar behaviour [3]. In figure 9, instantaneous vorticity isosurfaces for the SW and TW modes of this wake, computed at $Re = 220$, are displayed. Again, the TW state shown is a downwards-travelling wave, and, corresponding to the fact that in this case the critical Floquet multipliers do not lie so close to the negative real axis, the isosurfaces have a more oblique character. Once again, for the TW the streamwise isosurfaces on opposite sides of the wake interlace each other by $\lambda/4$.

The bifurcation diagram for the quasi-periodic modes of the square cylinder wake are shown in figure 10. In this case, the bifurcations are supercritical, and we have scenario 1 from figure 4. As for the circular cylinder case, the SW solutions are relatively unstable to the TW solutions.

TW Mode of the Driven Cavity

The vorticity dynamics of a spanwise-travelling mode can be seen in an example drawn from the computational study of the periodically driven cavity [3]. In figure 11 we see vorticity isosurfaces of the TW mode over one floor period, as seen from above the cavity. The wave travels via successive merging of braid vortices produced on opposite ($\pm y$) main rollers. As for the wake flows, the braid vortices produced on opposite sides of the cavity have an initial $\lambda/4$ interlacing. After period T has elapsed, the vortex structures are exactly the same as in the initial frame, but translated in the $+z$-direction — by an amount that can be derived from the secondary period, which in the TW case is related to wave propagation speed.

Conclusions

Quasi-periodic instability modes of time-periodic two-dimensional flows are just as generic as synchronous modes, and may in fact already have been observed in existing flows. This is possible because (*a*) in general, there is no physical necessity, even in single-parameter flows, that synchronous modes will be the first to become unstable and (*b*) the initially bifurcating modes do not necessarily remain dominant as the control parameter is increased. We have pointed out some of the salient features that enable TW modes to be discriminated from SW modes, and synchronous modes. When considering the computation of these flows, it is important to recognise that SW and TW modes belong to separate solution branches in the nonlinear case, and to check the relative stability of the two types of solution.

Acknowledgements

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