

Differential Diffusion: Often a Finite-Mixing Length Effect

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Abstract

Many instances of differential diffusion, i e, different species having different turbulent diffusion coefficients in the same flow, can be explained as a finite mixing length effect. That is, in a simple mixing length scenario, the turbulent diffusion coefficient

has the form $K = w_m l_m \left[1 + O\left(\frac{l_m}{L_c}\right)^2 \right]$ where, w_m is the mixing

velocity, l_m the mixing length and L_c the overall distribution scale for a particular species. The first term represents the familiar gradient diffusion while the second term becomes important when l_m/L_c is finite. This second term shows that different species will have different diffusion coefficients if they have different overall distribution scales. Such different L_c s may come about due to different boundary conditions and different intrinsic properties (molecular diffusivity, settling velocity etc) for different species. For momentum transfer in turbulent oscillatory boundary layers the second term is imaginary and explains observed phase leads of shear stresses ahead of velocity gradients.

Introduction

Understanding and modelling natural and industrial processes requires in very many cases understanding of turbulent mixing. The present paper addresses an aspect of turbulent mixing, which is still not well understood namely, differential diffusion. The term differential diffusion refers to the observation of different turbulent diffusion coefficients K defined by

$$K = - \frac{\text{turbulent mixing flux density}}{\text{concentration gradient}} \quad (1)$$

for different species in a given flow.

In physical oceanography, the species in focus are usually, momentum, heat and salt, and differential diffusion of these has been argued by Gargett [4] to be an important player in many oceanographical processes. Laboratory experiments, e g, Turner [13], Nagata & Komori [9] and Jackson & Rehmann [7] have shown differential diffusion of heat and salt. Turner [13] and Jackson & Rehmann [7] found $K_{\text{heat}} > K_{\text{salt}}$, and this has been rationalized, by Turner [13] and others, as being due to heat having a greater molecular diffusivity than salt. However, Nagata & Komori [9] found the opposite, i e $K_{\text{heat}} < K_{\text{salt}}$, in experiments which had a different geometry. Hence, differential diffusion cannot be totally due to differences in molecular diffusivity.

The idea that species with greater molecular diffusivity should diffuse more rapidly in turbulence has also been applied to various species in flame combustion by Kronenburg & Bilger [8]. Chanson ([1], [2]) has reported data, which show differential diffusion of momentum and bubbles in supercritical free surface flows. Differential diffusion of momentum and different sediment sizes has been documented for steady flows by Coleman [3] and Graf & Cellino [5] and for oscillatory flows by Nielsen [10] p 220.

Nielsen & Teakle [11] reconsidered the simple mixing length scenario of Figure 1, where parcels with vertical spacing l_m , the mixing length, are swapping positions traveling with vertical speed w_m , the mixing velocity.

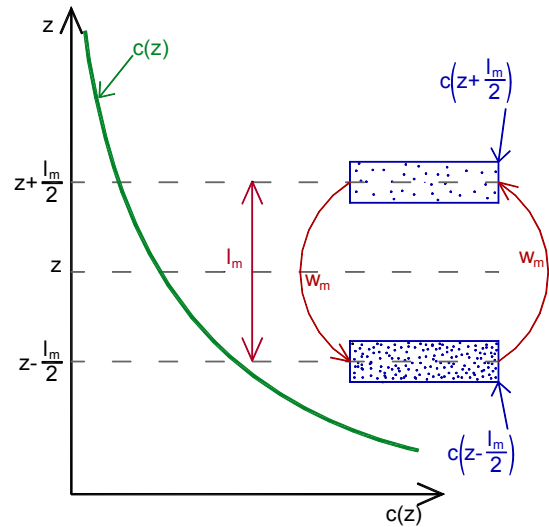


Figure 1: A simple mixing length scenario involving a single mixing length l_m and corresponding mixing velocity w_m .

The resulting flux density is then

$$\begin{aligned} q_m &= w_m \left[c\left(z - \frac{l_m}{2}\right) - c\left(z + \frac{l_m}{2}\right) \right] \\ &= -w_m l_m \frac{dc}{dz} \left[1 + O\left(\frac{l_m}{L_c}\right)^2 \right] \end{aligned} \quad (2)$$

which, by the definition (1), gives the diffusion coefficient

$K = w_m l_m \left[1 + O\left(\frac{l_m}{L_c}\right)^2 \right]$ of which the second term offers an explanation for differential diffusion as a finite mixing length effect. Examples are given below.

Steady Sediment Suspensions

Consider sediment particles with a range of settling velocities w_s suspended in the same turbulent flow, and adopting velocity fluctuations with the same statistics as those of the fluid. The turbulent swapping process in Figure 1 then creates an upward flux density given by (2), which in a steady situation is balanced by the settling flux $-w_s c(z)$ i e,

$$w_m \left[c\left(z - \frac{l_m}{2}\right) - c\left(z + \frac{l_m}{2}\right) \right] - w_s c(z) = 0 \quad (3)$$

In general (w_m , l_m) may be functions of z . However, the essence of differential diffusion as a finite mixing length effect is captured by

the simple case of homogeneous turbulence, i e, constant (w_m , l_m). In this case Equation (3) has solutions of the form

$$c(z) = C_0 e^{-z/L_s} \quad (4)$$

with

$$\begin{aligned} L_s &= \frac{l_m}{2 \sinh^{-1}\left(\frac{w_s}{2w_m}\right)} \\ &= \frac{l_m w_m}{w_s} \left[1 + \frac{1}{24} \left(\frac{w_s}{w_m}\right)^2 + \dots \right] \end{aligned} \quad (5)$$

The corresponding diffusion coefficient is given by

$$\begin{aligned} K_s &= \frac{q_m}{\frac{dc}{dz}} = \frac{w_m \left[c\left(z - \frac{l_m}{2}\right) - c\left(z + \frac{l_m}{2}\right) \right]}{-\frac{dc}{dz}} \\ &= w_m l_m \left[1 + \frac{1}{24} \left(\frac{w_s}{w_m}\right)^2 + \dots \right] \end{aligned} \quad (6)$$

This result quantifies differential diffusion for different sand sizes in homogeneous turbulence. It is however also in general qualitative agreement with data from river flows, where the turbulence is not homogeneous, see e g, Coleman [3], Graf & Cellino [5]. Indeed, van Rijn [14] suggested the empirical correction factor

$$\beta = \frac{K_s}{v_t} = 1 + 2 \left(\frac{w_s}{u_*}\right)^2 \quad (7)$$

for dealing with the observed differential diffusion of momentum (with diffusion coefficient v_t) and different sand fractions in natural rivers. Van Rijn's formula is seen to have the same dependence upon $w_s/w_m \sim w_s/u_*$ as the finite mixing length result (6).

We acknowledge that the example, which leads to (6), does not deal with all details, i e, in a natural scenario, particles with different settling velocities, and hence different response times w_s/g , do not get identical turbulent velocity statistics (\sim same w_m). However, this is usually totally overshadowed by the finite-mixing-length-effect. That is, the two act in opposite directions, but all the experimental data show K_s to be an increasing function of w_s as predicted by the finite-mixing-length model.

The importance of boundary conditions

The above example, of sediment particles with different w_s but statistically identical velocity fluctuations in homogeneous turbulence, can also be used to illustrate the importance of boundary conditions for the occurrence of differential diffusion. Consider point injections of such sediments into an infinite, homogeneous turbulence field, Figure 2. Each sediment type will then form a cloud, which sinks at an average rate of w_s . At the same time, each cloud grows in accordance with the theory of Taylor [12], i e, the vertical extent σ_z of each cloud grows in accordance with

$$\sigma_z(t) = \sqrt{2 w'^2 \int_0^t \int_0^\theta \rho_{ww}(\tau) d\tau d\theta} \quad (8)$$

where ρ_{ww} is the autocorrelation function for $w(t)$. This standard deviation, or cloud size, is independent of w_s under the assumption of identical fluctuation statistics. There is thus no differential diffusion in Taylor's infinite, unsteady scenario. Differential diffusion only occurs when a boundary condition, e g, $c(0,t) \equiv C_0$, as in the example above, is enforced.

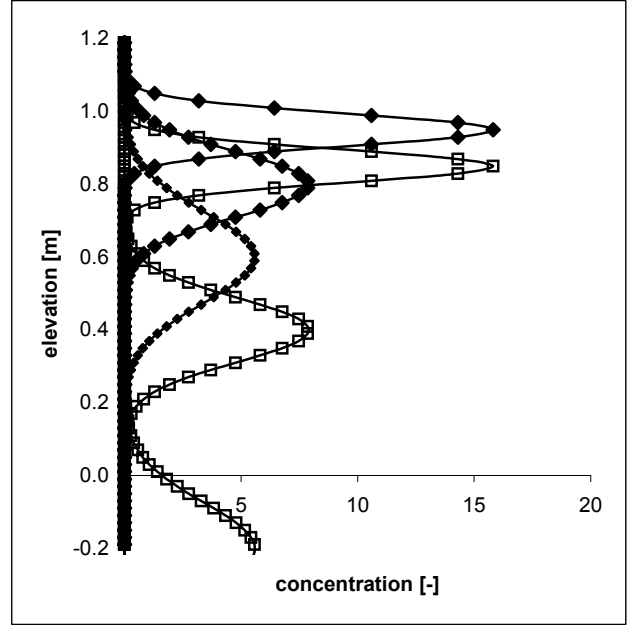


Figure 2: Successive concentration profiles $c(z,t)$, after 5, 10 and 20 seconds of particles with $w_s=1\text{cm/s}$ (\blacklozenge), and 3cm/s (\square) respectively settling with statistically identical velocity fluctuations after being released as point injections at $z=1\text{m}$ at $t=0$. At any time the concentration profiles have the same shape, irrespective of w_s .

Finite Mixing Length Effects on Eddy Viscosities

Steady current profiles, e g, the log-profile can be understood in terms of a mixing length model where only the first term (the gradient term) of (2) is retained. That is however not possible for oscillatory turbulent flows. The most obvious feature of oscillatory turbulent boundary layers, which requires finite mixing length terms, is the observed phase lead of local shear stresses ahead of the local velocity gradients, cf Figure 3.

In a formalism with real-valued eddy viscosities, shear stresses $\tau(z,t)$ being out of phase with the local velocity gradients $\partial u/\partial z$

leads, through the usual definition, $v_t = \frac{\tau/\rho}{\partial u/\partial z}$ to wildly

variable $[-\infty; \infty]$ eddy viscosities. Alternatively one can (for the simple harmonic case) use constant, but complex-valued eddy viscosities with argument equal to the phase lead of $\tau(z,t)$ relative to $\partial u/\partial z$. The complex-valued eddy viscosities are however just nominal and rather ad hoc tools. In the following, we make the case that the phase lead of $\tau(z,t)$ relative to $\partial u/\partial z$ is in fact a finite-mixing length effect. In this case the finite mixing length term in (2) is imaginary corresponding to the observed phase shift between $\tau(z,t)$ relative to $\partial u/\partial z$.

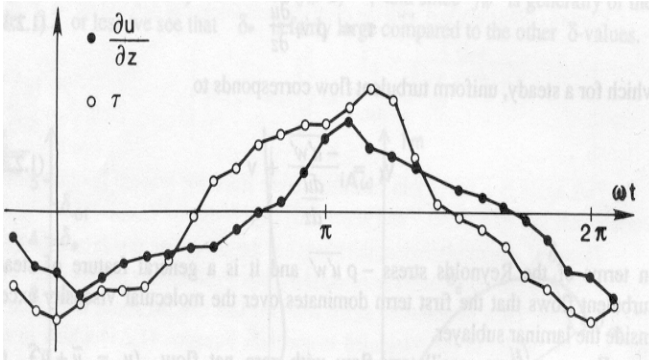


Figure 3: Velocity gradient and shear stress in a turbulent oscillatory boundary layer. The stress leads the velocity gradient. Measurements by Jonsson & Carlsen [6].

To show this we first note that, in the simple mixing length scenario in Figure 1 the upward flux of x -momentum ($= -\tau$) is

$$-\tau = w_m \left[\rho u \left(z + \frac{l_m}{2} \right) - \rho u \left(z - \frac{l_m}{2} \right) \right] \quad (9)$$

The velocity in a simple harmonic oscillatory boundary layer flow is often expressed in terms of the free stream velocity $u_\infty(t)$ and the complex velocity defect function $D(z)$ defined by

$$u(z, t) = A \omega e^{i\omega t} [1 - D(z)] = u_\infty(t) [1 - D(z)] \quad (10)$$

in terms of which the shear stress is

$$\frac{\tau(z, t)}{\rho} = \frac{\partial}{\partial t} \int_z^\infty [u_\infty(t) - u(z', t)] dz' = i \omega \int_z^\infty D(z') dz' \quad (11)$$

In combination with (9) this leads to the finite mixing length momentum equation

$$\frac{d}{dz} \left\{ w_m \left[D \left(z + \frac{l_m}{2} \right) - D \left(z - \frac{l_m}{2} \right) \right] \right\} = i \omega D(z) \quad (12)$$

This equation must, with the relevant expressions for $l_m(z)$ and $w_m(z)$ be solved with the boundary conditions $D(0) = 1$ and $D(\infty) = 0$. By letting l_m tend to zero, one sees that Equation (12) is the finite-mixing-length equivalent to the laminar

$$v \frac{d^2 D}{dz^2} = i \omega D \quad (13)$$

which, with the above mentioned boundary conditions has the solution

$$D = e^{-\sqrt{\frac{i\omega}{v}} z} \quad (14)$$

To see the essence of the finite mixing length effect on the momentum flux (9) and hence on v_i it suffices to consider the simplest scenario: homogeneous turbulence with constant (l_m, w_m) . In this case, the momentum equation (12) has an analogous solution to the " $l_m \rightarrow 0$ solution" (14).

Inserting a solution of this form, i.e., $D = e^{\alpha z}$, into (12) leads to the following expression for α

$$\alpha = -\sqrt{\frac{i\omega}{w_m l_m} \left(1 - \frac{1}{48} \frac{i\omega l_m}{w_m} - \dots \right)} \quad (15)$$

This expression for the exponent shows by comparison with the laminar solution (14) that the eddy viscosity is given by

$$v_t = w_m l_m \left(1 + \frac{1}{24} i \frac{l_m}{w_m / \omega} + \dots \right) \quad (16)$$

where the second, imaginary term shows that the phase lead of τ ahead of $\partial u / \partial z$ develops as l_m becomes large compared with the vertical scale $L_\omega = w_m / \omega \sim u_s / \omega$ defined by the mixing (friction) velocity and the wave frequency. For very slow oscillations the second term vanishes and the classical von Karman-Prandtl mixing length theory suffices.

Differential Diffusion at an Interface

Consider mixing at an interface as in Figure 4. The density variation may be due to temperature, salinity or suspended sediment. Horizontal fluid momentum may vary in a similar fashion at the interface.

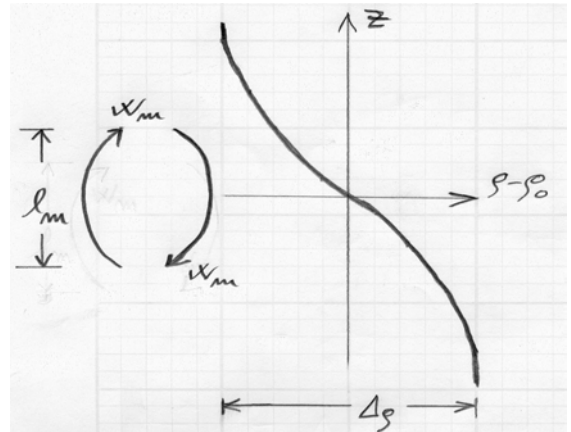


Figure 4: Assumed density distribution at an interface where turbulent mixing can be described by Equation (2) in terms of a single mixing length l_m and a single mixing velocity w_m .

Consider density as an example and, for simplicity, assume that the density profile is given by

$$\rho(z) = \rho_0 - \frac{\Delta \rho}{2} \tanh\left(\frac{z}{L}\right) \quad (17)$$

In this situation the mixing flux corresponding to the mixing length l_m and the mixing velocity w_m is

$$\begin{aligned} q_m &= w_m \left[\rho \left(z - \frac{l_m}{2} \right) - \rho \left(z + \frac{l_m}{2} \right) \right] \\ &= -w_m l_m \frac{d\rho}{dz} \left[1 - \frac{1}{12} \left(\frac{l_m}{L} \right)^2 \left[1 - 3 \tanh^2 \left(\frac{z}{L} \right) \right] \right] \end{aligned} \quad (18)$$

which for $z \ll L$ is simplified to

$$q_m \approx -w_m l_m \frac{d\rho}{dz} \left[1 - \frac{1}{12} \left(\frac{l_m}{L} \right)^2 \right] \quad (19)$$

The second term, the finite mixing length term, thus indicates greater diffusion coefficients for species with greater L . Equation (19) could thus explain the observed differential diffusion of salt and temperature with $K_s < K_t$ if the mixing length l_m is a finite fraction of the distribution scales and if the scale for the temperature L_t is greater than that of salt, L_s . In this case a positive feedback loop exists: Greater $L \Rightarrow$ greater $K \Rightarrow$ greater $L \dots$

Jackson & Rehmann [7] found that the differentiation disappears when the mixing gets very vigorous. That could also be explained by (19). That is, if the mixing is enhanced essentially by increasing w_m while l_m is more or less fixed. Both of L_t and L_s will then increase compared with l_m and the second term in the bracket (the finite mixing length effect) becomes insignificant.

Conclusions

In classical mixing length models, including those of Prandtl and von Karman, only the first term, the gradient term, is maintained in the Taylor expansion of the turbulent mixing flux:

$$\begin{aligned} q_m &= w_m \left[c \left(z - \frac{l_m}{2} \right) - c \left(z + \frac{l_m}{2} \right) \right] \\ &= -w_m l_m \frac{dc}{dz} \left[1 + O \left(\frac{l_m}{L_c} \right)^2 \right] \end{aligned} \quad (2)$$

This leads to $K = w_m l_m$ for all species in a given flow. In other words, differential diffusion is not anticipated by these models, at least not for species with the same w_m , l_m . On the other hand, differential diffusion is quite commonly observed even for species between which no significant differences in w_m , l_m are expected.

We have shown that keeping just one more term yields explanations for a number of conundrums related to turbulent mixing. That is, the second term in the mixing flux, which becomes important when l_m/L_c is finite, can explain that different sediment sizes suspended in the same flow display K -values which increase with increasing settling velocity.

For the mixing of momentum, salt and heat at an interface, Equation (2) also holds a possible explanation for differential diffusion. That is, if one species, for some reason, has a greater L_c , the finite mixing length terms will make it diffuse faster and thus provide a positive feedback loop for this difference.

While the classical theory of Taylor [12] does include finite mixing length effects in a Lagrangian sense, these are not expressed in terms of differential diffusion until steady boundary conditions are enforced.

For the transfer of momentum in turbulent oscillatory boundary layers the finite mixing length term is imaginary and explains observed phase leads of shear stresses ahead of velocity gradients.

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