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# Topographically trapped finite-amplitude kink solitons

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## Abstract

The evolution of internal wave fronts in an approximately linearly stratified fluid is considered where the fluid is contained in a channel with localized variations in width. It is assumed that there is an oncoming flow such that the long-wave speed of an internal wave mode is near zero. A variable coefficient finite-amplitude equation is presented which governs the wave evolution for uniform flows and rectangular channels, and it is shown that in these circumstances forced waves can be neglected. The interaction of kink soliton solutions of this finite-amplitude equation are considered.

## Introduction

In [2] the dynamics of weakly nonlinear internal wave fronts, or columnar disturbances, contained in a rectangular channel of variable width were considered in the transcritical limit (i.e. long-wave speed approximately zero), with uniform oncoming flow and weak contractions. There it was shown that the evolution was governed by a variable coefficient Korteweg–de Vries (KdV) equation, as the effective velocity experienced by the fronts accelerates within the contraction. The solutions of this equation for a step initial condition were considered in detail. For negative steps or rarefaction fronts it was demonstrated that steady solutions could evolve in the vicinity of the contraction which exhibited two types of hydraulic control: normal controls where the long-wave speed was zero at the maximum topographic perturbation and virtual controls where the long-wave speed was zero away from this point. For positive steps or compression fronts the interaction with the contraction was shown to be intrinsically unsteady, with solitary waves being generated on the downstream side of the contraction and propagating upstream.

The weakly nonlinear KdV model breaks down for two important circumstances; nearly linearly stratified fluids and two-layer fluids where the layer depth is approximately half the total depth. In the former case finite-amplitude effects must be included ([3]), and in the latter cubic nonlinearity resulting in an extended Korteweg–de Vries (eKdV) equation ([4]).

As solutions of the constant coefficient KdV equation with step initial conditions are unsteady, the interaction of fronts with a contraction in the weakly nonlinear KdV model must be unsteady and is dependent to some degree on the time that the front has evolved over before it reaches the contraction. To obtain steadily propagating front solutions in the long-wave limit requires at a minimum cubic nonlinearity, as opposed to the quadratic nonlinearity in the KdV equation. The eKdV equation has such steadily propagating step solutions, known as kinks or kink solitons or topological solitons. These kink solitons are also valid solutions for an approximately linearly stratified fluid. The investigation of kink solitons interacting with contractions is thus of significance for three reasons. Firstly, unlike finite-amplitude undular bores or rarefactions, the interaction of kink solitons has no weakly nonlinear analogue. Second, observations suggest that kink solitons, like non-topological solitons, appear to be robust nonlinear structures, and so their interaction with topography is of physical importance, especially in exchange type flows. Finally, kink solitons are steadily propagating structures and consequently their interaction can be considered independent of their starting position relative to the contraction.

In the next section the equations which govern the near-critical propagation of waves in a contraction are discussed. Forced waves are also generated for approximately linear stratification; it is shown that for rectangular channels and uniform oncoming flows these waves are of smaller amplitude and evolve on a longer timescale and hence can be ignored. In the final section solutions for kink solitons interacting with contractions are presented.

## Near-critical waves in a contraction

Consider the inviscid, nondiffusive flow of a stratified fluid in a channel. A Cartesian coordinate system $h(x,y,z)$ is introduced, where $h$ is the undisturbed height of the free-surface above the origin, $x$ is the horizontal coordinate along the channel, $y$ is the transverse coordinate and $z$ is the vertical coordinate, being positive upwards. The density is $\rho_0\rho(z-\zeta)$ where $\zeta$ is the nondimensional vertical particle displacement. The Boussinesq parameter, which characterizes the strength of the density perturbation, is then defined as $\beta = \rho(0)-\rho(1)$. The reduced gravity is then $g' = \beta g$, where $g$ is the acceleration due to gravity, and the normalized buoyancy frequency $N(z)$ is

$$N^2 = -\frac{\rho'}{\beta\rho}(=M). \tag{1}$$

The time is $(h/g')^{\frac{1}{2}}t$, while the fluid velocities are $(g'h)^{\frac{1}{2}}N_0 h(u,v,w)$, the free surface displacement is $eh\eta$, where the binary parameter $e$ is 0 for a rigid lid boundary condition and 1 for a free surface condition, and finally the pressure is $\rho_0 gh(P(z)+\beta p)$, where $P' = -\rho$.

The waves in the channel are assumed to have amplitude $O(\alpha)$ and wavelength $O(\mu^{-1})$. It is assumed that this is also the lengthscale of the perturbation of the contraction, which has amplitude $O(\varepsilon)$. For both $\mu$ and $\varepsilon$ it is assumed $\varepsilon, \mu \ll 1$. Therefore, introduce

$$(x,y,t) = \mu^{-1}(x',y',t'), \tag{2}$$

and define the side boundaries

$$b'_{\pm} = \pm b'_0(1+\varepsilon f_{\pm}). \tag{3}$$

The mean perturbation is

$$f = \frac{1}{2}(f_+ + f_-). \tag{4}$$

A mean flow $\bar{u}$ exists in the channel, which can be assumed to be independent of $x$ and $y$. At this point no other restrictions are placed on $f_{\pm}$ and $\bar{u}$. Hence,

$$(u,v,w) = (\bar{u}+\alpha u', \varepsilon v', \mu\alpha w'), \quad p = \alpha p', \quad \zeta = \alpha\zeta', \quad \eta = \alpha\beta\eta', \tag{5}$$

and primes are ignored hereafter.

**Forced waves**

As shown in [1] near criticality four cases of forced evolution equations apply in these circumstances:

**(F1)** If $\bar{u}$ and $f$ vary with height the appropriate choice for $\alpha$ and $\mu$ is

$$\alpha = \mu^2 = \varepsilon^{\frac{1}{2}}, \tag{6}$$

and no restriction is placed on $\beta$. The flow is assumed to be close to criticality, hence $\bar{u}$ can be written as

$$\bar{u} = U(z) - c + \varepsilon^{\frac{1}{2}}\Delta + O(\varepsilon), \tag{7}$$

where $c$ is the long-wave speed of the resonant mode and $\Delta$ is a detuning constant. A long timescale is introduced:

$$\tau = \varepsilon^{\frac{1}{2}} t, \tag{8}$$

and a perturbation solution for $\zeta$ is sought in $\varepsilon^{1/2}$ of the form

$$\zeta = A(x,\tau)\phi(z) + \varepsilon^{\frac{1}{2}}\zeta^{(1)} + O(\varepsilon), \tag{9}$$

where $\phi$ is the long-wave mode with speed $c$ for the shear $U(z)$. Then it can be shown that $A$ satisfies

$$A_\tau + \Delta A_x + rAA_x + sA_{xxx} = -F_x, \tag{10}$$

where the forcing function $F$ is proportional to an integral of $f$ over the height of the fluid and $r$ and $s$ are constants. The Boussinesq limit corresponds here to $\beta \to 0$. In general, this will only marginally affect the coefficients of (10). In this case, in terms of the original variables, the characteristic amplitude and timescale for the forced waves is $O(\varepsilon^{\frac{1}{2}})$ and $O(\varepsilon^{-\frac{3}{4}})$ respectively.

**(F2)** When $\bar{u}$ is constant with height and the stratification is uniform, i.e.

$$M = 1 + O(\beta), \tag{11}$$

the nonlinear term in (10) is $O(\beta)$, where typically $\beta \ll 1$, and the derivation of (F1) does not apply in this limit. This was alluded to in [1]; the appropriate choice of parameters in this case is

$$\alpha = 1, \quad \mu^2 = \varepsilon, \quad \beta = \sigma\varepsilon. \tag{12}$$

The expressions (7), (8) and (9) are again used with $\varepsilon^{1/2}$ replaced everywhere by $\varepsilon$ and the simplication that $U_z \equiv 0$. Then a similar evolution equation for $A$ to that derived by [3] will result. The nonlinear term in this case is $O(\sigma)$, therefore in the Boussinesq limit $\sigma \to 0$, with all else unchanged, it would be expected that any forced wave will grow without being limited by nonlinear effects until it overturns. In this case, in terms of the original variables, the characteristic amplitude and timescale for the forced waves is $O(1)$ and $O(\varepsilon^{-\frac{3}{2}})$ respectively.

**(F3)** In the joint limit when $\beta \to 0$ and $\bar{u}$ and $f$ are independent of height the forcing term in (10) disappears. However, it was demonstrated in [1] that resonance will still occur due to Boussinesq effects. Here

$$\alpha = \mu^2 = \varepsilon, \quad \beta = \sigma\varepsilon, \tag{13}$$

and the same expressions for $\bar{u}$, $\tau$ and $\zeta$ are introduced as for (F2). Subsequently it can be shown that $A$ satisfies

$$A_\tau + (\Delta A)_x + rAA_x + sA_{xxx} = \gamma\Delta_x. \tag{14}$$

The coefficient of the forcing term, $\gamma$, is $O(\sigma)$, clearly in the Boussinesq limit the forced wave vanishes. In this case, in terms of the original variables, the characteristic amplitude and timescale for the forced waves is $O((\beta\varepsilon)^{\frac{1}{2}})$ and $O(\beta^{-1}\varepsilon^{-\frac{1}{2}})$ respectively.

**(F4)** Finally, for uniform stratification the nonlinear term in (14) vanishes. To derive a forced nonlinear evolution equation the appropriate choice of parameters is now

$$\alpha = \mu^2 = \varepsilon^{\frac{1}{2}}, \quad \beta = \sigma\varepsilon^{\frac{1}{2}}. \tag{15}$$

The same expressions for $\bar{u}$ and $\tau$ are introduced as in (F3), however as the expansion for $\zeta$ occurs in powers of $\varepsilon^{1/2}$ (9) is introduced. The perturbation expansion must be taken to second order to demonstrate that $A$ again satisfies a version of (14). For this equation the coefficient of both the nonlinear term, $r$, and the forcing, $\gamma$ are $O(\sigma)$. In this case, in terms of the original variables, the characteristic amplitude and timescale for the forced waves is $O(\varepsilon^{\frac{1}{2}})$ and $O(\beta^{-1}\varepsilon^{-\frac{3}{4}})$ respectively. In the limit $\sigma \ll 1$ forced waves will still be generated, however they take an infinitely long time to form.

Other types of resonance can occur in a contraction apart from the generation of forced waves. For example shear fronts propagating towards a contraction can become trapped and resonate. The analysis of (14) in the limit $\gamma \to 0$ in [2] clearly demonstrates this. For each of the above circumstances we can therefore consider the amplitude of the quasi-steady solution which form in a contraction due to this interaction and the timescale for its formation. In (F1) and (F2) the amplitudes and timescales are identical to the forced waves, therefore the only effect of a shear front will be to perturb one quasi-steady solution of (10) to another quasi-steady solution. However, for (F3) the amplitude and timescales are respectively $O(\varepsilon)$ and $O(\varepsilon^{-\frac{3}{2}})$, while for (F4) they are $O(\beta^{-1}\varepsilon)$ and $O(\varepsilon^{-\frac{5}{4}})$. In these latter two cases, since typically $\varepsilon \gg \beta$ the amplitudes for these unforced waves are larger than the corresponding forced waves, while the timescales the unforced waves form over are smaller than those for the forced waves. This suggests that the forced waves studied in the cases (F3) and (F4) are only relevant if there are no unforced waves in the vicinity of the contraction, in particular, when these unforced waves are shear fronts. In conclusion, if there are shear fronts in the vicinity of the contraction, for (F1) and (F2) these can be ignored, while for (F3) and (F4) the forced waves can be ignored. Thus we are led to consider the appropriate evolution equations for unforced waves.

**Unforced waves**

Two equations describe the general evolution of unforced waves in these circumstances. The appropriate equation is dependent on the stratification:

**(U1)** In the limit $\sigma \to 0$ of (F3), which corresponds to making the Boussinesq approximation, the forcing disappears and the amplitude satisfies the variable coefficient KdV equation

$$A_\tau + (\Delta A)_x + rAA_x + sA_{xxx} = 0. \tag{16}$$

This equation was studied in detail in [2]. In particular it was shown that in the limit $r \to 0$ and for long contractions the amplitude could grow without bounds if the sign of the $\Delta$ changed within the contraction. This leads us to consider the appropriate evolution equation for uniform stratification, for which $r = 0$ in the Boussinesq limit.

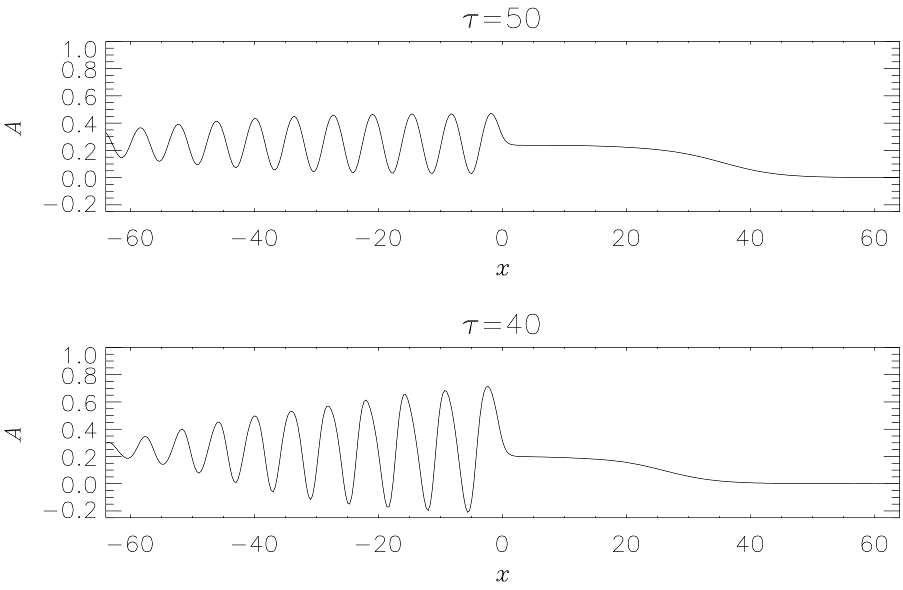


Figure 1: Interaction of a finite-amplitude kink soliton in a contraction demonstrating the effect of changing the width of the contraction. In both panels $q = -2$, $r = \frac{3}{4}$ and $\Delta_0 = 1$. For the upper panel $\Delta_1 = \frac{1}{2}$ and for the lower panel $\Delta_1 = 1$.

**(U2)** Case (F4) suggests that resonant shear fronts will attain an asymptotic amplitude $O(\beta^{-1}\varepsilon)$. Therefore, since typically $\beta \ll \varepsilon$ finite-amplitude effects must be considered. The derivation of the appropriate equation is summarized here. Introducing the same scalings as in (F2), a perturbation solution for $\zeta$ is sought of the form

$$\zeta = cA(x,\tau)\phi + \varepsilon\zeta^{(1)} + O(\varepsilon^2), \tag{17a}$$

where

$$\phi = \sin n\pi z, \quad c = \frac{1}{n\pi}, \tag{17b}$$

and $n$ is a positive, nonzero integer. Define $z(\xi, A)$ to be the solution of

$$\xi = z - \frac{A}{\pi}\sin\pi z, \tag{18}$$

which has a unique solution if $|A| < 1$. If this condition is violated then the wave has overturned, however providing it is satisfied then the amplitude satisfies the integro-differential equation

$$\int_\infty^x K(A,A')A'_\tau dx' + \Delta A + \frac{c^3}{2}(\sigma m(A) + A_{xx}) = 0, \tag{19a}$$

where

$$\Delta = \hat{\Delta} + cf, \tag{19b}$$

$$K(A,A') = \pi^2 \int_0^1 \frac{\partial z}{\partial A}\left[\frac{\partial z'}{\partial A'}\left(1 + \frac{\partial z'}{\partial \xi}\right) - (z - z')\frac{\partial}{\partial \xi}\left(\frac{\partial z'}{\partial A'}\right)\right] d\xi, \tag{19c}$$

$$m(A) = 2\pi^4 \int_0^1 (z - \xi)X_{eff}(\xi)z_A \, d\xi + \frac{A^2}{3}(1 - (-1)^n) - e\left(2A - (-1)^n 3A^2 + A^3\right), \tag{19d}$$

and for $n > 1$ $X_{eff}$ is a smoothed version of the buoyancy frequency perturbation:

$$X_{eff}(\xi) = \sum_{k=0}^{n-1} X\left(\frac{1}{n}\left(k + \frac{1}{2}(1 - (-1)^k) + (-1)^k\xi\right)\right). \tag{20}$$

**Finite-amplitude waves.**

We now consider finite-amplitude solutions for kink solitons in a contraction; however before proceeding a few comments can be made concerning (19). Equation (19) can be normalized by introducing

$$\tau^* = \alpha^{-\frac{3}{2}}\frac{c^3\tau}{2}, \quad \Delta^* = \frac{2\alpha\Delta}{c^3}, \quad \sigma^* = \alpha\sigma, \quad x^* = \alpha^{-\frac{1}{2}}x, \tag{21}$$

where $\alpha$ is some positive, arbitrary constant. Then, dropping the asterisks, the amplitude satisfies

$$\int_\infty^x K(A,A')A'_\tau dx' + \Delta A + \sigma m(A) + A_{xx} = 0. \tag{22}$$

Consider the term $\sigma m(A)$ of (22). Let

$$X(z) = az - b. \tag{23}$$

Thus

$$m(A) = \frac{A}{c^2}\left(\frac{a}{2} - b\right) + \frac{A^2}{3}(1 - 4a)(1 - (-1)^n) - e(2A - (-1)^n 3A^2 + A^3), \tag{24}$$

which, as noted in [3], contains no terms of order greater that cubic. The linear terms of $\sigma m(A)$ can be incorporated in the velocity, hence $\sigma m(A)$ can in general be written as

$$\sigma m(A) = \frac{1}{2}rA^2 + \frac{1}{3}qA^3, \tag{25}$$

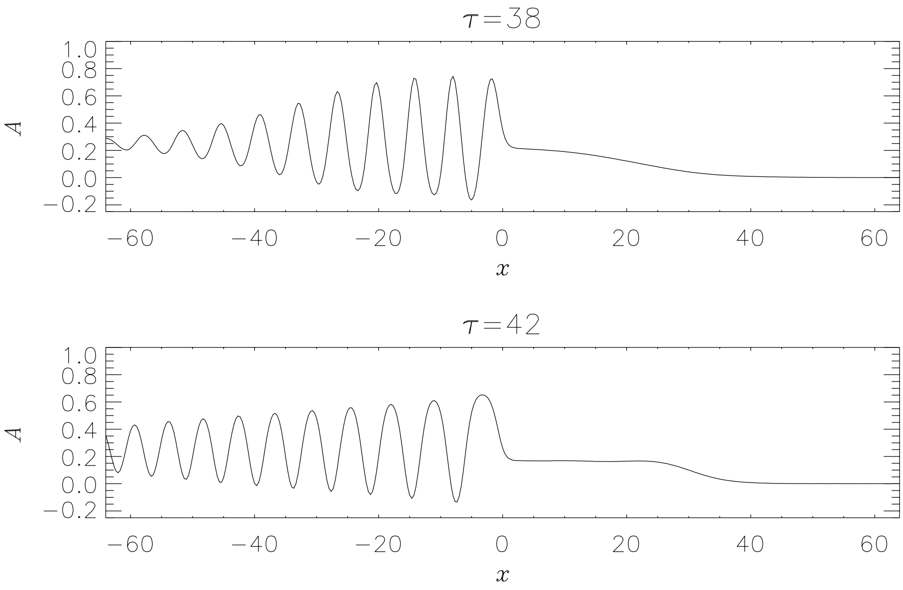


Figure 2: Interaction of a finite-amplitude kink soliton in a contraction demonstrating the effect of changing the strength of the stratification. In both panels $\Delta_0 = 1$ and $\Delta = \frac{1}{2}$. For the upper panel $q = -1$ and $r = \frac{3}{8}$, while in the lower panel $q = -4$ and $r = \frac{3}{2}$. The upper panel of figure 1 is intermediate between these two figures.

where, the only limitation at this point is that $q = -3e\sigma \leq 0$. For $\Delta \equiv 0$, (22) has kink soliton solutions (see [5]):

$$A = \frac{\Lambda}{2}(1 - \tanh\kappa(x - V\tau)), \tag{26a}$$

where

$$\Lambda = -\frac{2r}{3q}, \quad \kappa = \left(-\frac{q\Lambda^2}{8}\right)^{\frac{1}{2}}, \quad V = \frac{r\Lambda}{3}. \tag{26b}$$

Thus we assume an initial condition

$$A = \frac{\Lambda}{2}(1 - \tanh\kappa(x - x_0)), \tag{27}$$

where $x_0 < 0$ and $\Lambda$ and $\kappa$ are as specified by (26). The velocity perturbation is assumed to be here of the form:

$$\Delta(x) = \Delta_0 - \Delta_1 \operatorname{sech}^2(x/x_a), \tag{28}$$

and by an appropriate definition of $\alpha$ in (21) we can assume $x_a = 1$. The results presented here are largely independent of the exact form of the topography, provided it is smoothly varying with a single extrema and characteristic lengthscale.

Figures 1 and 2 show some selected solutions which demonstrate significant effects of the interaction of kink solitons with a contraction. The system which is solved is (19) with (25), (28) and (27). The numerical simulations use a synthesis of the methods used by [3] and [2]. This involves differentiating (19), resulting in a Volterra integral equation of the second type for $A_\tau$ at each time step. Spatial derivatives are evaluated by removing the step in $A$ and using pseudospectral methods for the residual function.

In figure 1 the effect of changing the width of the contraction is shown. For narrow contractions, as typified by the upper panel, the kink soliton is able to pass through the contraction with an insignificant reduction in the amplitude. After the generation of transient waves a steady, constant amplitude lee wave then forms downstream of the contraction. For larger contractions the transmitted front is reduced correspondingly. Further simulations suggest that for the parameters of figure 1 there is no transmitted wave for $\Delta_1 \geq 3$. Downstream of the contraction it appears that no steady lee wave train has formed, even though the upstream behaviour is no longer transient. Shortly after the time shown the simulation was halted due to overturning.

The effect of changing the stratification strength is shown in figure 2. As with figure 1 the transmitted front decreases as this effect strengthens. Secondly, increasing nonlinearity dampens the growth of the downstream lee wave. For weak stratification a constant amplitude lee wave starts to evolve just prior to the simulation being halted, while for the stronger stratification it appears that a downstream plateau starts to evolve which at large times would result in a steady hydraulically controlled solution in the vicinity of the contraction.

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### References


[1] S.R. Clarke and R.H.J. Grimshaw. Resonantly generated internal waves in a contraction. *J. Fluid Mech.*, 274:139–161, 1994.

[2] S.R. Clarke and R.H.J. Grimshaw. Weakly nonlinear internal wave fronts trapped in contractions. *J. Fluid Mech.*, 415:323–345, 2000.

[3] R. Grimshaw and Z. Yi. Resonant generation of finite-amplitude waves by the flow of a uniformly stratified fluid over topography. *J. Fluid Mech.*, 229:603–628, 1991.

[4] T.R. Marchant and N.F. Smyth. The extended Korteweg-de Vries equation and the resonant flow of a fluid over topography. *J. Fluid Mech.*, 221:263–288, 1990.

[5] J.W. Miles. On internal solitary waves. *Tellus*, 31:456–462, 1979.