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Spin-up flow of an Incompressible Fluid

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Abstract

Spin-up flows of an incompressible homogeneous fluid have been reviewed. Characteristics of spin-up are followed by a summary of well-established previous papers along the orders : linear spin-up, weakly non-linear spin-up and non-linear spin-up. Discussions are given to open problems from previous analytic theories as comparing with full numerical solutions for Navier-Stokes equation.

Introduction

Spin-up refers to the transient adjustment process of a confined fluid when the rotation rate of the container undergoes a change. Specifically, consider an incompressible viscous fluid which fills a closed circular cylindrical container [radius R, height H, aspect ratio $H/R \sim O(1)$]. At the initial state, both the fluid and the cylinder are in rigid-body rotation about the longitudinal axis (z-axis) at rotation rate Ω_i. At time $t=0$, the rotation rate of the cylinder is increased abruptly to $\Omega_f \left(\equiv \Omega_i + \Delta\Omega\right)$. The transient motion of the fluid, in response to this abrupt alteration of the rotation rate of the container, constitutes the spin-up. For the majority of geophysical and technological applications, the flow is characterized by the smallness of the Ekman number E [$\equiv \nu/\Omega_f H^2$, where ν is the kinematic viscosity of fluid]. The condition $E \ll 1$ leads to a boundary-layer-type global flow field, which consists of the inviscid interior, the Ekman and Stewartson boundary layers close to the solid walls of the container.

The pioneering treatise of Greenspan and Howard[11] considered the linear spin-up problem in which the Rossby number $\varepsilon = \Delta\Omega/\Omega_f$ is small. By undertaking detailed theoretical analyses, it was shown that spin-up is substantially accomplished over the spin-up timescale $E^{-1/2}\Omega_f^{-1}$, which is an order-of-magnitude smaller than the diffusive timescale $E^{-1}\Omega_f^{-1}$. The main ingredients of the adjustment process are the inviscid-boundary layer interactions, together with the angular momentum conservation in the inviscid interior. The essential flow character has since been verified by numerical simulations and laboratory experiments [see review articles by Benton & Clark [4], Duck & Foster [7]]. For definitiveness, the cylinder is oriented such that the central longitudinal axis is aligned in the vertical (z) axis, and the horizontal direction refers to the radial (r) and azimuthal (θ) directions.

Efforts were made to extend the linear problem formulation of Greenspan and Howard [11] to nonlinear settings. For ε small, but finite, Greenspan and Weinbaum [12] carried out series expansions using ε to include higher-order terms, which rendered a weakly nonlinear theory. Wedemeyer [33], in a departure from the approach of Greenspan and Weinbaum [12], delineated the character of spin-up flow from the initiate state of rest, i.e., $\Omega_i = 0$, $\varepsilon = 1.0$. When the container starts rotating from rest, Wedemeyer's solution demonstrated the existence of the radially-propagated velocity shear front. The interior inviscid fluid at smaller (larger) radii than the shear front moves toward (away from) the horizontal boundary layers. Venezian [28,29] generalized the analysis by Wedemeyer to the range $0 \ll \varepsilon < 1.0$. It is important to point out the difficulties associated with the above-stated flow models. The theory of Greenspan & Weinbaum [12] does not recover the nonlinear flow of Wedemeyer in the limit $\varepsilon \to 1.0$. Similarly, the solutions of Wedemeyer and Venezian do not reproduce completely the linear results of Greenspan & Howard as $\varepsilon \to 0$. These are indicative of the challenges ahead in unifying the linear and nonlinear solutions.

In 1970's and 1980's, numerical solutions to the full Navier-Stokes equations were obtained [16, 18, 32]. These endeavors verified the global flow field predicted by the previous theoretical studies. Further extensions were made by numerical simulations to explore spin-up with a free surface [5, 20, 21], turbulent flows[6], in a geostrophic flow [21], to name a few. In 1990's, different aspects of spin-up, somewhat modified from the classical models, were investigated. Examples include spin-up in a non-axisymmetric container [14, 26]. Also, spin-up flows were delineated when the shapes of the solid walls of the container were deformed. In summary, these are representative of the efforts to tackle more realistic situations.

In the present review, spin-up flows of an incompressible fluid will be dealt with. Spin-up of a stratified fluid and/or a compressible fluid is a separate topic, and a considerable body of research has been accumulated [2, 15, 17, 25, 30].

The longstanding problem areas in classical spin-up research may be summarized:

(a) Nonlinear Ekman compatibility condition [19, 34],

(b) Unifying the prediction of linear and nonlinear flow models [12, 28, 29],

(c) On the viscous effects near the cylindrical sidewall [3, 19].

General Problem Formulation

As stated previously, at the initial state, the rotation rate of the fluid and the cylindrical container is Ω_i, and the rotation rate of the cylinder is increased instantaneously to Ω_f [see figure 1]. The Rossby number, $\varepsilon \equiv \Delta\Omega/\Omega_{ref}$, where $\Delta\Omega = \Omega_f - \Omega_i$, is a measure of nonlinearity. Here, the reference rotation rate, Ω_{ref}, is either Ω_f or Ω_i. For the nonlinear spin-up, it is convenient to set $\Omega_{ref} = \Omega_f$, and $\Omega_i = (1-\varepsilon)\Omega_f$, as done in [28, 33]. In the case of a linear problem, it is customary to select $\Omega_{ref} = \Omega_i$, $\Omega_f = (1+\varepsilon)\Omega_i$, as in [11, 12]. In the present paper, $\Omega_{ref} = \Omega_f$, unless otherwise noted.

Nondimensionalization is made of the physical quantities of interest:

$$(r^*, z^*) = (r/H, z/H)\ ,\ a^* = R/H\ ,\ t^* = t\Omega_{ref}\ ,$$

$$\vec{V}^* = \frac{\vec{V}}{\Omega_{ref} H}\ ,\ p^* = \frac{p}{\rho(\Omega_{ref} H)^2}\ ,$$

in which superscript * represents nondimensional quantities, t time, $\vec{V}(\equiv (u,v,w))$ velocity vector in the (radial, azimuthal, axial) coordinate, ρ fluid density, and p pressure.

The governing time-dependent axisymmetric Navier-Stokes equations, in nondimensional form, are straightforward, after dropping *:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + E\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{u}{r}\right) + \frac{\partial^2 u}{\partial z^2}\right), \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = E\left(\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{\partial^2 v}{\partial z^2}\right), \quad (2)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + E\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right). \quad (3)$$

In the above, the Ekman number $E \equiv \nu/\Omega_{ref} H^2$.

The associated initial and boundary conditions are stated:

$$u = v - (1-\varepsilon)r = w = 0 \ \text{ at } \ t = 0\,; \quad (4a)$$

$$u = v - r = w = 0 \ \text{ at } \ z = \pm 1/2\,; \quad (4b)$$

$$u = v = \partial w/\partial r = 0 \ \text{ at } \ r = 0\,; \quad (4c)$$

$$u = v - a = w = 0 \ \text{ at } \ r = a\,. \quad (4d)$$

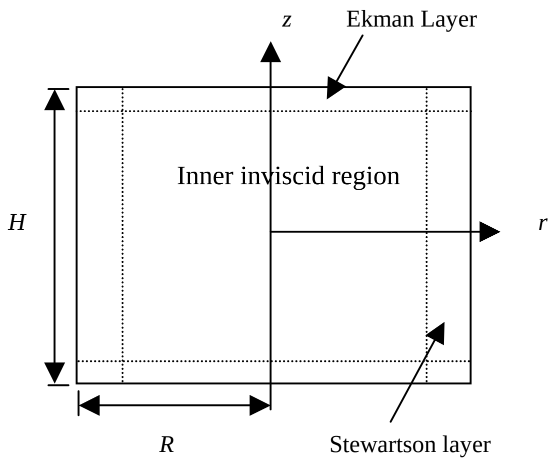

Figure 1. Coordinates and flow regime in the meridional plane

Flow Regime

By inspecting the governing equations, the fact $E << 1$ leads to the assertion that the overall flow field is of boundary-layer-type. It is advantageous to depict the character of the individual flow region [see figure 1].

Horizontal Boundary Layers

The horizontal boundary layer, often termed the Ekman layer, forms near the (horizontal) endwall disks of the container. The fluid in this layer is propelled radially-outward due to the increased rotation rate ($\Omega_i + \Delta\Omega$). This, in turn, causes suction in the axial direction of the inviscid interior fluid, which is known as the Ekman pumping. The crux of the argument is that, by way of the Ekman pumping, the rotating disk exerts control of the meridional (secondary) flow. The thickness of Ekman layer is scaled $O(E^{1/2})$, as can be inferred from the governing equations.

Vertical Boundary Layer

The horizontal-propelled fluid flux in the Ekman layer, after reaching the vertical sidewall, is carried vertically along the cylindrical sidewall. Obviously, in order to meet the no-slip condition at the sidewall, a boundary layer is called for. This vertical boundary layer, which is termed the Stewartson layer, is to satisfy the no-slip condition for both the azimuthal and vertical directions. It has been established that the vertical layer consists of a double-layer structure, i.e., $O(E^{1/4})$-layer for the azimuthal flow, and $O(E^{1/3})$-layer for the vertical flow. It should be mentioned that, in linear spin-up, the vertical boundary layer is less important in controlling the interior flow. However, in nonlinear spin-up, the influence of vertical layer is not insignificant in the determination of the interior fluid motion [1, 13].

Inviscid Interior Region

For $E << 1$, the thicknesses of the above-stated boundary layers are thin and the bulk of the flow domain is essentially inviscid. In this region, the principal force balance is between the pressure gradient and Coriolis force. Also, the radial and azimuthal velocities are substantially uniform in the axial direction, i.e., the Taylor-Proudman column is maintained [8,9,10]. Thus, by treating the axial variations of flow properly, there is a possibility that the (r,z,t) dependent three-dimensional problem can be reformulated as a (r,t) dependent two-dimensional problem in the inviscid region.

Linear Spin-up

Consideration is given to the transient flow when the rotation rate of the cylinder changes abruptly from Ω_i to $\Omega_f\left[\equiv (1+\varepsilon)\Omega_i\right]$. The problem is linearized under the assumption that the Rossby number $\varepsilon << 1$. In the original treatise of Greenspan and Howard [11], the governing equations were linearized under $E << 1, \varepsilon << 1$, and an analytical solution was secured by means of the Laplace transform. It should be pointed out that, in Greenspan & Howard [11], the reference rotation rate of the problem was chosen to be Ω_i, not Ω_f.

In a short duration $t \sim O(\Omega_i^{-1})$ after the impulsive start-up of the cylinder, the Ekman boundary layer of thickness $O(E^{1/2})$ is formed near the endwall disk of the cylinder. A complete mathematical solution is available for the linear Ekman layer, and the scales are $u \sim O(\varepsilon), v \sim O(\varepsilon), w \sim O(\varepsilon E^{1/2})$. Also, the axial variations of (u,v) are well-known spiral structure, as demonstrated in [10].

In the Ekman layer, the mass flux $O(\varepsilon E^{1/2})$ of fluid is propelled radially outward, this sucks in the fluid in the inviscid interior. This creates an axial flow, $w \sim O(\varepsilon E^{1/2})$, which is termed the Ekman pumping, toward the disk. In the inviscid interior, a concomitant radial flow, $u \sim O(\varepsilon E^{1/2})$, is induced. Therefore, in the inviscid interior, in order to satisfy angular momentum conservation, the angular velocity increases by way of vortex stretching, i.e., the spin-up process takes place. The bulk of interior fluid undergoes the spin-up process, and, therefore, the overall adjustment process is substantially accomplished over the spin-up timescale $O(E^{-1/2})$, not over the diffusion time scale $O(E^{-1})$.

The radial distance (d) that a fluid parcel has travelled over the spin-up process is very small, $O(\varepsilon)$. This can be inferred from the fact that the distance is the product of the radial velocity scale $u \sim O(\varepsilon E^{1/2})$ and the spin-up time scale $O(E^{-1/2})$.

The time-dependent velocity fields over the spin-up time scale (in the inertial frame) are described approximately [11]:

$$v = r + \varepsilon S(2t) r \left(1 - \exp(-E^{1/2} t)\right), \qquad (5)$$

$$\Psi / \varepsilon = -rzE^{1/2} S(2t) \exp(-E^{1/2} t) . \qquad (6)$$

In which Ψ denotes the meridional stream function $\left(u = \partial\Psi/\partial r, w = -\partial\Psi/\partial z\right)$, and $S(t)$ the Fresnel integral,

$$C(t) + iS(t) = \int_0^t (2\pi z)^{-1/2} \exp(iz) dz . \qquad (7)$$

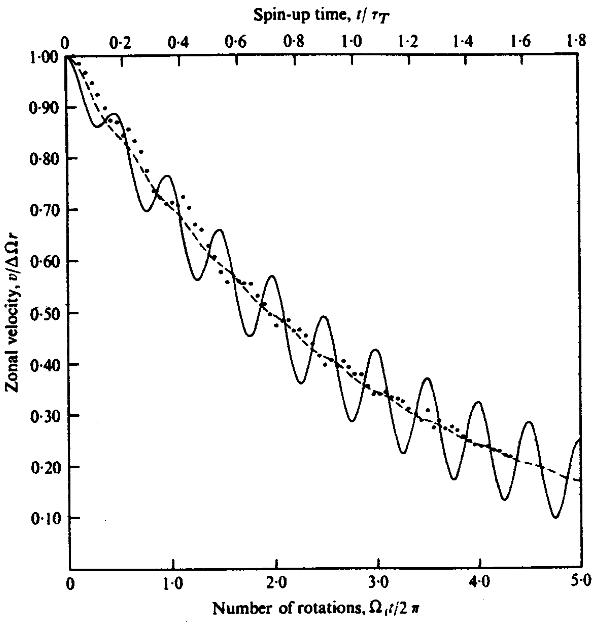

Figure 2. Comparison of the experiment results ($E = 3.30 \times 10^{-3}, \varepsilon = 0.111$) with the analytical results of [11], • • • , experimental data [31], - - - - - - - , theory [eq.(3.18) of [11]], ―――――, approximate solution [eq.(5)], (printed from [31]).

The theoretical depiction of linear spin-up has since been validated by a multiple scaling analysis [28] as well as experiments [32] and numerical simulation [15] [see, see figure 2].

It is to be noted that the spin-up theory of Greenspan & Howard was developed under the assumption of two infinite disks. This also reinforces the observation that the role of the cylindrical sidewall on the spin-up of interior fluid is insignificant. As remarked previously, the distance that a fluid percel moves in the linear spin-up is d, which is much smaller than the thickness of vertical sidewall boundary layer $O(E^{1/4})$, i.e., $d \sim O(\varepsilon) << O(E^{1/4})$. It implies that over the duration of spin-up, the fluid originating in the sidewall layer stays in this layer; the influence of the sidewall boundary layer is meager on the interior fluid. For the present closed container, however, the vertical boundary layer exists, and the structure of this layer has been a subject of another investigations [1, 14].

Weakly Non-linear Spin-up

For ε small, but finite, Greenspan & Weinbaum [12] devised a nonlinear model from the afore-described linear model. They expressed the flow variable by series expansions of the Rossby number ε. In the course of analysis, due to the presence of the secular terms, the strained coordinate method by Poincare and Lighthill was introduced. Due to the complication in the analysis, expansions were obtained up to the second-order, i.e., $O(\varepsilon^2)$. The result for the azimuthal velocity is shown :

$$v = r + \varepsilon r \left(1 - \exp(-2\bar{t})\right) + O(\varepsilon^2), \qquad (8)$$

in which $$E^{1/2} t = \bar{t} + \varepsilon \left(-\frac{1}{2}\bar{t} + \frac{2}{5}(1 - \exp(-2\bar{t})) \right). \qquad (9)$$

This weakly nonlinear model of Greenspan & Weinbaum was shown to be consistent with the experiments in the range $|\varepsilon| < 0.5$.

Specifically, two issues are to be stressed:

(1) Because of the nonlinear effect, in comparison to the linear theory, the time to reach the final state becomes shorter (longer) in spin-up (spin-down). Since the theory neglects the higher-order terms, it is not clear if the same conclusion is applicable when $\varepsilon \sim O(1)$. However, the later experiments and computations seem to be in support of the qualitative validity of the theory up to $\varepsilon \sim O(1)$ [34]. It will be a challenge to supplement the theory to produce a more versatile analytical model [see figure 3].

(2) As stated earlier, the theory of Greenspan and Weinbaum is for two infinite disks with no sidewall. Consequently, this theory is incapable of predicting the presence of the propagating velocity shear front which is a hallmark of nonlinear spin-up in a confined closed container. This is perhaps a more serious drawback of the theory [28, 29].

Non-linear Spin-up

Wedemeyer Model ($\varepsilon = 1.0$)

An elegant theoretical model was put forth by Wedemeyer [33] about spin-up from rest ($\varepsilon = 1.0$). In rapidly-rotating flow, the well-known axial uniformity of horizontal velocities in the inviscid interior (Taylor-Proudman column, i.e., $\partial u/\partial z = \partial v/\partial z = 0$) is exploited. A simplified equation for v is obtained:

$$\frac{\partial v}{\partial t} + u\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) = E\left(\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right). \qquad (10)$$

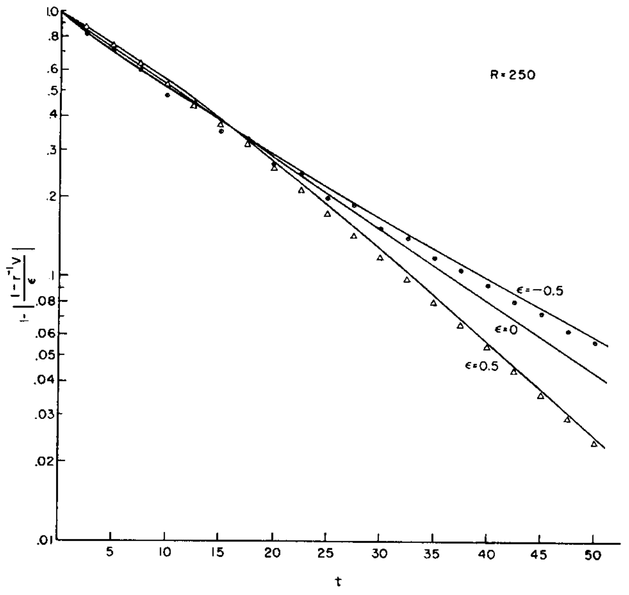

Figure 3. Comparisons between theory (——) of [11] and numerical data (Δ, ■) , (printed from [12]).

In order to solve the above equation, a relationship linking u and v is necessary. To this end, Wedemeyer made use of the numerical result of Rogers & Lance [23]. Furthermore, noting the inter-relations between the inviscid interior flow and the Ekman layer flow, Wedemeyer made the following assumptions:

(a) the boundary layer flow is quasi-steady;

(b) the finite geometry of the cylinder does not effect the boundary layer flux;

(c) the inerior fluid is in rigid-body rotation.

Combining the above assumption and the local similarity assumption based on the Rogers & Lance data, Wedemeyer arrived at an approximate relationship linking u and v :

$$u = E^{1/2}(v - r) . \tag{11}$$

Substituting eq.(11) into eq.(10) yields

$$\frac{\partial v}{\partial \tau} + (v - r)\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) = E^{1/2}\left(\frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right). \tag{12}$$

Under $E << 1$, dropping the viscous terms in eq.(12) leads to

$$\text{(i)}\ \ v = 0, w = 2E^{1/2} z , \tag{13a}$$

for the region ahead of the shear front, i.e., $r \le ae^{-\tau}$

$$\text{(ii)}\ \ v = \frac{1}{1-\exp(-2\tau)}\left(r - \frac{a^2}{r}\exp(-2\tau)\right),\ \ w = -\frac{2E^{1/2} z}{\exp(2\tau)-1} , \tag{13b}$$

for the region behind the shear front, i.e., $r \ge ae^{-\tau}$, where $\tau = E^{1/2} t$.

The Wedemeyer solution, Eq. (13a-b), successfully depicts much of the prominent flow characteristics. The existence of the propagating shear front and the global flow field and other nonlinear behavior are captured well in this model. However, inconsistencies are discernible as well. For instance, the behavior of the Ekman pumping velocity w in both sides of the front is at variance with assumption (i). Also, the adoption of the numerical data of Rogers & Lance poses serious difficulty. According to the data of Rogers and Lance, the Ekman pumping takes place from the region of small angular velocity to that of large angular velocity. However, the Wedemeyer solution shows that, in the inviscid region behind the front ($r \ge ae^{-\tau}$), the Ekman pumping is directed from the disk-region with a higher rotation rate to the inviscid-region with a lower rotation rate. These, and others, illustrate conflicting assumptions and solutions. Despite these shortcomings, the Wedemeyer solution provides overall flow pictures which are shown to be generally compatible with the numerical solutions [16, 18, 32]. The inconsistencies of Wedemeyer model have been pointed out[3, 16], but little serious efforts have been made to improve the fundamental foundation of the model.

Extension of the Wedemeyer Model ($0 < \varepsilon \le 1.0$)

Expanding upon the approaches of Wedemeyer, Venezian [28-29] produced an analytical solution for the spin-up from a finite rotation rate ($\Omega_i \ne 0$) to another finite rotation rate ($\Omega_f \ne 0$) for $0 < \varepsilon \le 1.0$:

$$v = \frac{r}{1 + \frac{\varepsilon}{1-\varepsilon}\exp(-2\tau)} \tag{14a}$$

at $r \le a(1 - \varepsilon + \varepsilon\exp(-2\tau))^{1/2}$,

$$v = \frac{1}{1-\exp(-2\tau)}\left(r - \frac{a^2}{r}\exp(-2\tau)\right) \tag{14b}$$

at $r \ge a(1 - \varepsilon + \varepsilon\exp(-2\tau))^{1/2}$.

The theoretical solution of Venezian demonstrates the salient nonlinear features for $\varepsilon \sim O(1)$, which include the spin-up from rest. However, difficulties are encountered as this solution attempts to recover the linear spin-up by letting $\varepsilon << 1$. In addition, it is unclear to come up with a physically convincing explanation about Wedemeyer's assertion that the propagating shear front represents the propagating of characteristic line.

One contribution of Venezian's model lies in the improved treatment of viscous terms in the vicinity of the front. Venezian, by a careful analysis of the thin viscous layer in the neighborhood of the shear front, gave a description of flow surrounding this front:

$$v = 4E^{1/4}(2\pi\eta)^{-1/2}(ar\exp(\beta^2)erfc(\beta))^{-1} , \tag{15}$$

in which

$\eta = \exp(2E^{1/2}t) - 1$, $\beta^2 = a^2(r^2\exp(2E^{1/2}t) - 1)^2 / 8E^{1/2}\eta$, and a being nondimensional radius of the container.

The analytical solution of Venezian was shown to be generally compatible with the full Navier-Stokes numerical solutions, as displayed in figure 4. Clearly, the viscous solution of Venezian, in comparison to the Wedemeyer inviscid solution, is in better agreement with the numerical results. However, the discrepancies near the shear front are not in substantial.

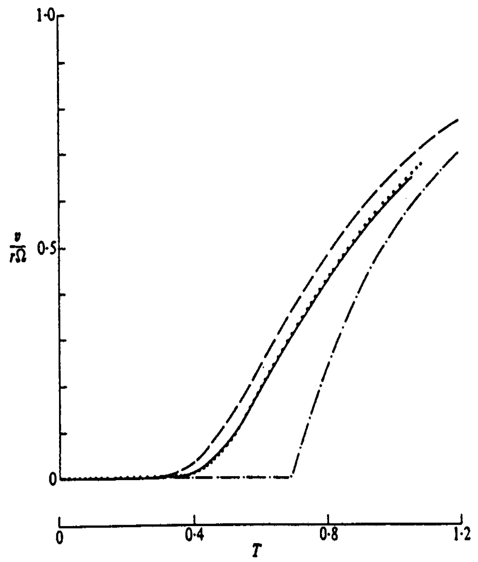

Figure 4. Time histories of azimuthal velocity ($E = 9.18 \times 10^{-5}$). The vertical location is at mid-depth. The radial location is 0.5.
—·—·—·—·—· , Wedemeyer's inviscid solution;
———————— , numerical results [16];
.... , LDV measurements;
- - - - - - - - - - - - , Venezian's profile [eq.(15)],
(printed from [16]).

The Weidman Model

Weidman [34] noted that the linear Ekman pumping condition was the source of inconsistencies. From that standpoint, Weidman attempted to use a 7th-degree polynomial curve fitting to the numerical data of Rogers & Lance [23] [see figure 5].

$$\frac{\partial v}{\partial \tau} + F(v)\frac{\partial (rv)}{\partial r} = 0\,, \tag{16}$$

in which

$$F(s) = 4.8867s^7 - 21.06589s^6 + 38.23662s^5 - 38.45380s^4$$

$$+ 23.88193s^3 - 9.44264s^2 + 1.07258s + 0.88446\,.$$

Weidman [34] proceeded to make numerical integration and the results contained physically unrealistic flow descriptions near the shear front. In this connection, Benton [3] ascertained an inherent difficulty with the local similarity assumption in utilizing the data of Rogers & Lance [23]. The current trend is to find supplementary ideas to improve or refine the basic model of Wedemeyer. The recent endeavors by Konijnenberg & Heijst [19] also ascertained the need to modify the modeling of Ekman pumping.

Concluding Remarks

The backbone of the modern study of spin-up is the Greenspan & Howard model of linear spin-up ($\varepsilon << 1, E << 1$). The role of Ekman layer pumping, together with the meridional circulation, is clearly captured in the model. The overall transient process is accomplished over the spin-up time scale $O(E^{-1/2}\Omega_f^{-1})$.

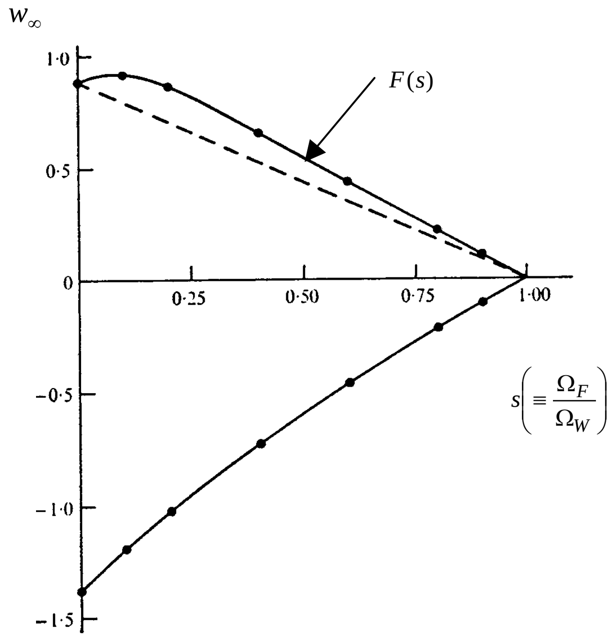

Figure 5. Nonlinear Ekman pumping velocity. w_∞ is Ekman pumping velocity at the boundary layer edge , Ω_F angular velocity of the far-field fluid and Ω_W angular velocity of the disk wall, (printed from [34]).

The expansion of the model to nonlinear ranges is a challenging issue. The limitation of the weakly nonlinear model is pointed out. The success and shortcomings of the Wedemeyer model ($\varepsilon = 1.0$) are re-visited. The attempts to expand upon or improve this model are discussed.

The classical topic of spin-up is still alive, and some of the future directions are listed.

References

[1] Barcilon, V., Stewartson Layers in transient rotating fluid flows, *J. Fluid Mech.*, **33**, 1968, 815-825.
[2] Bark, F.H., Meijer, P.S. & Cohen, H.I., Spin-up of a rapidly rotating gas, *Phys. Fluids*, **21**(4), 1978, 531-539.
[3] Benton, E.R., Vorticity dynamics in spin-up from rest, *Phys. Fluids*, **22**, 1979, 1250-1251.
[4] Benton, E.R. & Clark, J., Spin-up, *Ann. Rev. Fluid Mech.*, **6**, 1974, 257-280.
[5] Cederlof, U., Free-surface effects on spin-up, *J. Fluid Mech.*, **187**, 1988, 395-407.
[6] Cui, A. & Street, R.L., Large-eddy simulation of turbulence rotating convective flow development, *J. Fluid Mech.*, **447**, 2001, 53-84.
[7] Duck, P.W. & Foster, M.R., Spin-up of homogeneous and stratified fluids, *Ann. Rev. Fluid Mech.*, **33**, 2001, 231-263.
[8] Greenspan, H.P., On the transient motion of a contained rotating fluid, *J. Fluid Mech.*, **20**, 1964, 673-696.
[9] Greenspan, H.P., On the general theory of contained rotating fluid motions, *J. Fluid Mech.*, **22**, 1965, 449-462.
[10] Greenspan, H.P., *The theory of rotating fluids*, Cambridge Univ. Press, 1968.
[11] Greenspan, H.P. & Howard, L.N., On a time dependent motion of a rotating fluid, *J. Fluid Mech.*, **17**, 1963, 385-404.
[12] Greenspan, H.P. & Weinbaum, S., On nonlinear spin-up of a rotating fluid, *J. Math. & Phys.*, **44**, 1965, 66-85.

[13] Heijst, G.J.F.van, The shear layer structure in a rotating fluid near a differentially rotating sidewall, *J. Fluid Mech.*, **130**, 1983, 1-12.
[14] Heijst, G.J.F. van, Spin-up phenomena in non-axisymmetric container, *J. Fluid Mech.*, **206**, 1989, 171-191.
[15] Hyun, J.M., Fowlis, W.W. & Warn-Varnas, A., Numerical solutions for spin-up of a stratified fluid, *J. Fluid Mech.*, **117**, 1982, 71-90.
[16] Hyun, J.M., Leslie, F., Fowlis, W.M. & Warn-Varnas, A., Numerical solutions for spin-up from rest in a cylinder, *J. Fluid Mech.*, **127**, 1983, 263-281.
[17] Hyun, J.M. & Park, J.S., Spin-up from rest of a compressible fluid in a rapidly rotating cylinder, *J. Fluid Mech.*, **237**, 1992, 413-434.
[18] Kitchen, C.W.J., Navier-Stokes for spin-up in a filled cylinder, *AIAA J.*, **18**(8), 1980, 929-934.
[19] Konijnenberg, J.A. van de & Heijst, G.J.F. van, Nonlinear spin-up in a circular cylinder, *Phys. Fluids*, **7**(12), 1995, 2989-2999.
[20] Maars, L.R., Nonlinear and free-surface effects on the spin-down of batropic axisymmetric vortices, *J. Fluid Mech.*, **246**, 1993, 117-141.
[21] O'Donnel, J. & Linden, P.F., Free-surface effects on the spin-up of fluid in a rotating cylinder, *J. Fluid Mech.*, **232**, 1991, 439-453.
[22] Reznik, G.M. & Grimshaw, R., Nonlinear geostrophic adjustment in the presence of a boundary, *J. Fluid Mech.*, **472**, 2002, 257-283.
[23] Rogers, M.H. & Lance, G.N., The rotationally symmetric flow of a viscous fluid on the presence of an infinite rotating disk, *J. Fluid Mech.*, **7**, 1960, 617-631.
[24] Saffman, P.G. & Baker, G.R., Vortex interactions, *Ann. Rev. Fluid Mech.*, **11**, 1979, 95-122.
[25] Sakurai, T. Spin-down problem of rotating stratified fluid in thermally insulated circular cylinders, *J. Fluid Mech.*, **37**, 1969, 689-699.
[26] Suh, Y.K. & Choi, Y.H., Study on the spin-up of fluid in a rectangular container using Ekman pumping models, *J. Fluid Mech.*, **458**, 2002, 103-132.
[27] St-Maurice, J.P. & Veronis, G., A multiple-scaling analysis of the spin-up problem, *J. Fluid Mech.*, **68**, 1975, 417-445.
[28]Venezian, G., Spin-up of a contained fluid, in *Topics in ocean engineering* editors C.I. Bretschneider, Gulf Publishing Co., 1969, 212-223.
[29] Venezian, G., Nonlinear spin-up, in *Topics in ocean engineering* editors C.I. Bretschneider, Gulf Publishing Co., 1970, 87-96.
[30] Walin, G. Some aspects of time-dependent motion of a stratified rotating fluid, *J. Fluid Mech.*, **36**, 1969, 289-307.
[31] Warn-Varnas, A., Fowlis, W., Piacs S. & Lee, S.M., Numerical solutions and laser-Doppler measurements, *J. Fluid Mech.* **85**, 1978, 609-639.
[32] Watkins, W.B. & Hussey, R.G., Spin-up from rest: limitations of the Wedemeyer model, *Phys. Fluids*, **16**(9), 1973, 1530-1531.
[33] Wedemeyer, E.H., The unsteady flow within a spinning cylinder, *J. Fluid Mech.*, **20**, 1964, 383-399.
[34] Weidman, P.D., On the spin-up and spin-down of a rotating fluid. Part1. & 2., *J. Fluid Mech.*, **77**, 1976, 685-735.