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Exchange flow between continuously stratified reservoirs

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Abstract

We provide a solution to inviscid steady exchange flow between continuously stratified reservoirs, where it is assumed that the flow in each direction is independently self-similar. The solution requires knowledge only of the two reservoirs stratifications and an imposed net barotropic through flow, and includes regions of stagnant fluid which separate two counter-flowing, stably stratified layers. It is argued that these stagnant, or inactive layers, may play a role in oceanic and geophysical flows, but are inherently difficult to observe in field measurements.

Introduction

Density driven exchange flow through a channel connecting two reservoirs occurs in many geophysical systems, in particular between ocean basins, between semi-enclosed seas and the open ocean, and at the mouth of estuaries. To first order, a solution for bi-directional exchange can be obtained by assuming a two-layer structure in the flow [1, 2, 7]. The two-layer hydraulic solution can be used to give a first order prediction of flow through a channel, assuming that the end reservoirs are homogeneous and that the fluid is inviscid, hydrostatic, non-rotating and incompressible. However geophysical flows are invariably more complicated than the two-layer theory. Here we present the results of a study into the influence of stable stratification in the reservoirs upon exchange flows.

Continuously stratified internal hydraulics was investigated by Killworth [5] for the case of uni-directional flow. In appendix A of that paper it was demonstrated that if inviscid bi-directional exchange between stratified reservoirs were to occur, then the vertical position of the streamline dividing the two counter-flowing regions can only occupy one position in the vertical: it must be flat everywhere. This result breaks down in two cases; first, when the density coordinates are discontinuous (that is, when the vertical gradient in density is zero) and second, when there is a discontinuity in density at the dividing streamline.

Engqvist [3] used a multi-layer formulation to include stratification into bi-directional exchange flows through a contraction. The multiple layers are divided into left- and right-flowing groups and if these two groups of active layers are separated by a central layer which has zero velocity, then the problem can be solved. The central stagnant layer has the effect of decoupling the two groups of layers so that control conditions for each group of layers independently satisfies Wood's [6] criteria for control of the selective withdrawal problem.

Motivated by the result of Killworth [5] we have investigated the problem of flow through a flat-bottomed contracting channel using analytical techniques and a two-dimensional numerical model. We extend Engqvist's [3] layered solution to continuously stratified flow. In addition we show a solution where the two active layers are allowed to make contact at one point. We proceed test this theory against the numerical model.

Analytical model

The selective withdrawal problem

The derivation of selective withdrawal of an inviscid fluid from a stratified reservoir was originally due to Wood [6]. One assumes a single layer flow in which the reservoir density profiles and channel shape is known. The linear Bernoulli function for a Boussinesq fluid is written

$$B(x,z) = \frac{p+\rho g z}{\rho_0}, \tag{1}$$

where x is the horizontal coordinate, z denotes height, g acceleration due to gravity, ρ density, ρ_0 the reference density and p pressure. Define η, the upstream height coordinate which follows streamlines (so that in the reservoir $z = \eta$), and note that density conservation implies that $\rho = \rho(\eta)$ only. Conservation of energy along a streamline is then simply

$$\tfrac{1}{2}u^2 + B = B_\infty, \tag{2}$$

where u is horizontal velocity and $B_\infty(\eta)$ is the Bernoulli function in the upstream reservoir (where $u = 0$).

Conservation of volume along a streamline can be written

$$ubz_\eta = Q(\eta), \tag{3}$$

where $b(x)$ is the channel width and Q is the flux along the streamline. Equations (2) and (3) then allow solution of the selective withdrawal problem.

These equations are now applied to the withdrawal of a stably stratified fluid (with total upstream depth h) through a contracting channel. The boundary conditions on the flowing layer include a solid surface on the lower boundary where $z = 0$, and a free upper surface with a density jump $\delta\rho$. We assume that flow within the layer is self-similar [6], or in other words the height and energy of a streamline can be separated into x- and η-dependent parts:

$$z(x,\eta) = \alpha(x)\eta, \tag{4}$$

$$B(x,\eta) = \alpha(x)B_\infty(\eta). \tag{5}$$

The factor $\alpha(x)$ describes the reduction in height of a streamline from the upstream reservoir conditions.

The assumption of self-similarity allow us to write down a solution to this problem. The horizontal velocity u can be deduced from (2), and substituted into (3) to obtain

$$b(x)\alpha(x)(1-\alpha(x))^{1/2} = \frac{Q(\eta)}{(2B_\infty(\eta))^{1/2}}. \tag{6}$$

The left hand side of (6) depends only upon x, while the right hand side is a function of η, implying that both sides are constant. The x-derivative of (6) gives

$$\frac{1}{b}\frac{db}{dx} = \frac{1}{\alpha}\frac{d\alpha}{dx}\left(\frac{3\alpha-2}{2(1-\alpha)}\right). \tag{7}$$

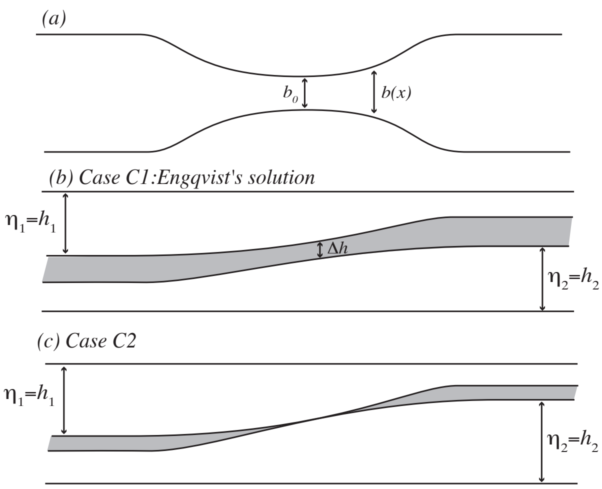

Figure 1: Schematic of stratified exchange flow. (*a*) Contracting channel in plan view; (*b*) case C1; Engqvist's solution for two decoupled layers separated by a minimum distance of Δh; (*c*) case C2; solution where flowing layers touch at one (and only one) point. In both cases upstream stratification for each layer, and upstream height coordinate are known.

When there is a minimum in $b(x)$, then either $d\alpha/dx = 0$ or $\alpha = 2/3$. The latter case was defined by Wood [6] as a point of hydraulic control, which enables us to find velocity and density everywhere provided that the withdrawal height h is known.

Application of self-similar flow to bi-directional exchange

We now apply Wood's self-similar solution to exchange flows, where we have two stably stratified reservoirs at either end of a channel which has a simple minimum in width. Two cases are considered, as depicted in figure 1. In both examples we allow for two active layers flowing in opposite directions, and use two upstream vertical coordinates (η_1 for the upper layer, and η_2 for the lower layer). The first scenario (case C1, figure 1(*b*)) is simply the continuous extension of the layered solution proposed by Engqvist [3], and we therefore refer to it as Engqvist's solution. In this solution there are two active layers which are divided by a stagnant, or inactive region in which we assume that vertical gradients of both velocity and density are zero. At some point in the channel the stagnant region has a minimum thickness which we call Δh. For this case we expect the density jump which bounds the flowing layers, $\delta\rho$, to be zero.

In the second case (C2, figure 1(*c*)), the two flowing layers touch at a single point. This is analogous to Engqvist's solution with $\Delta h = 0$, and $\delta\rho$ finite. The inactive region still exists in this solution, and is of finite thickness throughout the length of the channel except for the point where the layers touch.

The solution of these equations requires knowledge of the upstream withdrawal height h_i for each layer. We have several restrictions which allow us to calculate these heights, namely that the two layers are closest at some point x_0

$$\alpha_1(x_0)h_1 + \alpha_2(x_0)h_2 = 1 - \Delta h, \tag{8}$$

(where all heights have been nondimensionalised by the total height H of the channel). In addition there are conditions on the density of the streamlines bounding the active layers, namely

$$(\mathrm{C1}): \qquad \rho_1(h_1) = \rho_2(h_2) \qquad \delta\rho = 0, \tag{9a}$$

$$(\mathrm{C2}): \qquad \Delta h = 0 \qquad \delta\rho = \frac{\rho_2(h_2) - \rho_1(h_1)}{2}, \tag{9b}$$

where we have assumed that stagnant fluid has density which is the half-way between the densities on the streamline bounding either active layer.

One more condition on layer heights is required to close this problem. This condition is a specified barotropic flux q_0 through the channel. For the cases shown here, we retain symmetry, so that $q_0 = 0$, and reservoir stratifications are linear and symmetric (although more general solutions are possible via iterative methods [4]). Because of this symmetry, the point x_0 is found at the centre of the channel, $\alpha_i(x_0) = \frac{2}{3}$, $h_1 = h_2$ and (8) becomes

$$\Delta h = 1 - \tfrac{4}{3}h_i. \tag{10}$$

This is sufficient to close the problem.

Comparisons

We now test the theory (and, by implication, the self-similar assumption) by simulating stratified exchange flows with a hydrostatic 2-dimensional numerical model (see [4] for details). We compare the numerical and analytical solutions, to determine whether the assumption of self-similarity and the derived solution provide a suitable description of stratified exchange flows. In comparing the two solutions, we are primarily interested in whether the self-similar solution is selected by the numerical simulation, but note that we expect to see some differences due to the role of diffusion and viscosity in the simulations. Thus we anticipate that the numerical solution will be more diffuse, particularly close to the edge of the active layers, where discontinuities in velocity and density gradients are present in the analytical model.

In the two cases presented, both reservoirs are linearly stratified with the same density gradient (described by the top to bottom density difference δ_V), but the mean density of each reservoir is offset by a small amount (the horizontal density difference being δ_H). The ratio of vertical to horizontal density differences $r_\rho = \frac{\delta_V}{\delta_H}$ governs whether solution C1 or C2 will be selected. For large values of r_ρ the stratification in the reservoirs is strong and solution C1 is expected. For small r_ρ solution C2 is possible.

Case C1

The analytical flow field is shown in figure 2(*a*) for the case $r_\rho = 4$. The numerical simulation, shown in panel (*b*), appears to be qualitatively similar; both solutions show anti-symmetric layers which accelerate through the contraction. A region of zero (or small) velocities divides the two flowing layers. The two lower panels in this diagram show the differences between the numerical and analytical solutions for the density (*c*) and velocity (*d*) fields as a percentage of the range in density and velocity respectively. These diagrams show that, to first order, the assumption of self-similarity applies in the numerical solution. Within the active layers the differences between the two solutions are generally very small ($<3\%$). Self-similarity deteriorates with distance along the channel, which is expected from the impact of diffusion and viscosity in the numerical solution.

The largest differences between the numerical and analytical solutions (10–15%) can be seen at the edge of the flowing layers, and within the inactive regions. The analytical solution predicts a discontinuity in the velocity gradient at the edge of the flowing layers, so that the diffusive flux in the numerical solution is expected to be large. Viscosity thus acts to thin the stagnant region in the centre of the channel (near $x/L = 0$) so that it might appear from the numerical solution that the stagnant region does not extend to the centre of the channel. The velocit-

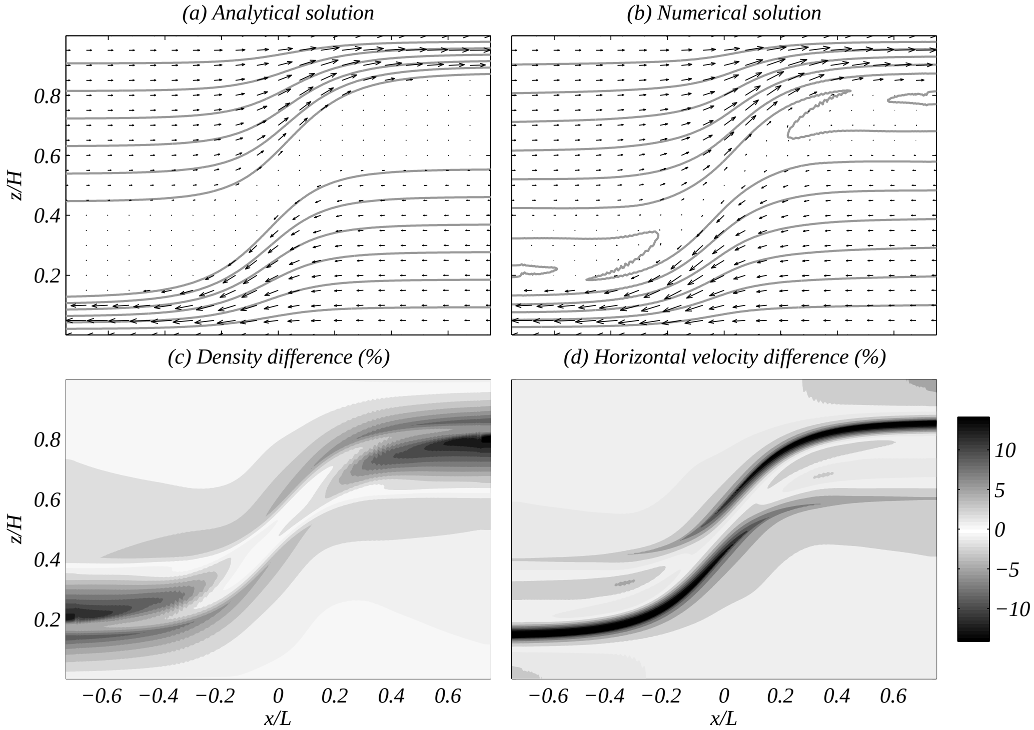

Figure 2: Engqvist's solution (case C1) with $r_\rho = 4$ and $q_0 = 0$. Isopycnals (contours) and velocity (vectors) for (*a*) the self-similar analytical solution; (*b*) numerical simulations. (*c*) Difference in density and (*d*) difference in horizontal velocity.

ies in this region are small compared to velocities in the active layers, but the effects of recirculation can be seen in the density field. In fact, the largest errors in the density field are due to these recirculations, which produce a statically unstable density profile in the inactive parts of the flow. These unstable regions are allowed to develop, as the numerical convection routine is turned off. It is shown in Hogg and Killworth [4] that the addition of convection removes the unstable density fields without altering flow in the active layers.

Case C2

A more stringent test of the self-similarity assumption is the case where stratification is further weakened, as shown in figure 3 so that the analytical solution requires that the layers meet at a point and the density jump, $\delta\rho$ is finite. The implication of this solution is that some fluid from (say) reservoir 2 is dense enough that it might be exchanged, but that the requirement for self-similar flow, in combination with the existence of flow in layer 1, acts to block the passage of this dense fluid. The data shown in figure 3 supports the hypothesis that self-similar flow occurs in this case. This can be seen by the small differences ($< 5\%$) within the two flowing layers. In this case there are discontinuities in both density and velocity at the edges of the flowing layers in the analytical solution. The result is that diffusion is large in the numerical solution so that differences between numerical and analytical predictions are as great as 20% at the edges of the layers, and also acts to reduce the total transport of each layer.

The analytical solution in figure 3 includes a finite width stagnant region everywhere except at $x = 0$. However the numerical solution (figure 3(*b*)) shows that the stagnant region is only observable there towards either end of the channel. This highlights the role of viscosity in thinning the stagnant region. One can infer from these simulations, which use the minimum viscosity necessary for stability, that stagnant layers are unlikely to be observed in geophysical flows with reasonable values of viscosity. Nonetheless, the importance of self-similarity in the simulated flow indicates the relevance of the solution presented here as an estimate of exchange flux.

Discussion

We have presented a method of calculating flow between stratified reservoirs assuming that flow within each of two active layers is self-similar. The simplification yielded by the assumption of self-similarity allows us to solve analytically for flow in both layers, producing a solution which predicts that parts of the fluid column are inactive. It is these inactive or stagnant regions which allow us to overcome the paradox raised by Killworth [5]. In that paper it was shown that if bi-directional stratified exchange flow were to occur, then the zero-velocity streamline can only occupy one vertical position. The self-similar solutions bypass this condition because the zero streamline(s) occur in regions of the flow where there is no vertical gradient in density.

We have proposed two self-similar cases which are solutions for stratified bi-directional flow through a flat-bottomed contracting channel. Confirmation of the solutions are difficult. The theory described here is inviscid by necessity, and yet we are using it as a theoretical model to describe flows which will always feel the effect of viscosity. This is true for observed geophysical flows, laboratory models and numerical simulations. Numerical simulation allows the greatest scope for comparison, since we can run the model such that viscosity and diffusion are minimised. These comparisons show the first order effect that viscosity and diffusion have upon the flow, and that the major elements of the theoretical solution can be identified in the simulated flows.

In the examples shown, the inactive regions are thinned by

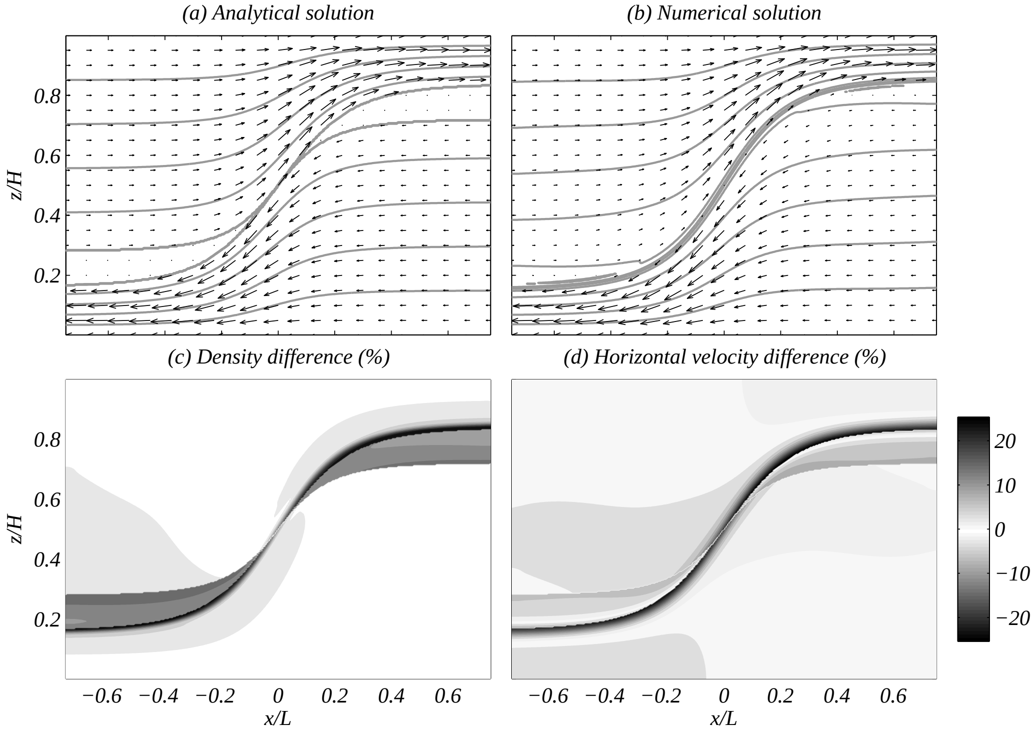

Figure 3: Case C2 using $r_\rho = 1$ and $q_0 = 0$. Panels as for figure 2.

the transfer of momentum from the neighbouring active layers. This acts to mask the existence of the stagnant regions in the viscous case, and provides an explanation as to why the self-similar solution to stratified exchange flows has not been considered before as a general solution to this problem. There is unlikely to be any observational evidence indicating that a stagnant region plays a role in dynamic flows.

The stagnant layers which are specified in these analytical solutions present two difficulties. Firstly there is the question of how they form, and secondly how they connect to reservoir conditions. The former question presents little difficulty for case C1, where the stagnant layer is constrained in density by the fact that the bounding densities of the two active layers are equal. However, for case C2 we have (somewhat arbitrarily) chosen the stagnant layers to have density which is intermediate between the two bounding densities. It is important to stress that this density cannot be achieved by mixing in our inviscid solution. We have chosen this density as a suitable boundary condition on the active layers which allow computation of the solution.

Similarly, it is not possible for the homogeneous stagnant layers to connect smoothly to reservoir densities in a time dependent flow for either cases C1 or C2. A possible scenario is that convection (which is not included in these simulations) is important. One might expect this to alter the solution to some extent, however in hydraulically controlled flows such as these the solution is controlled at the throat of the contraction; so that the most relevant quantity is the density which acts as a boundary condition on the active layers there. For this reason the assumed homogeneous stagnant layers produce a solution which closely approximates the time dependent cases.

It may also be noted that the fast flowing layers do not match reservoir conditions. In realistic flows it would be expected that a transition to subcritical flow (probably via an internal hydraulic jump) would occur. We have omitted this possibility by ignoring hydraulic jumps in the analytical solution, and by setting the boundary conditions at the open boundaries of the numerical solution to allow supercritical flow to pass out of the model domain. Internal hydraulic jumps may well occur in more realistic flows, however the presence of stagnant layers means that the jump is insulated from the neighbouring active layer.

Despite these unresolved issues, the solutions presented here provide a simple resolution to a problem which has been assumed in the past to be too complicated to address. The solutions give useful and simple estimates for flux through a contraction in a channel between stratified reservoirs, and the problem can be solved analytically.

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