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2D Velocity-Field Analysis Using Triple Decomposition of Motion

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Abstract

A straightforward application of the new triple decomposition of the local relative motion near a point to 2D velocity fields is presented. Unlike the Cauchy-Stokes double decomposition of motion into a pure straining motion and a rigid-body rotation the triple decomposition of motion aims, moreover, at the extraction of an *effective* pure shearing motion. It results in a more detailed flow description as the third term is responsible for a specific portion of vorticity labelled "*shear* vorticity" and for a specific portion of strain rate labelled "*shear* strain rate".

Introduction

The concept of the triple decomposition of motion has been motivated by a longstanding problem of a vortex and vorticity expressed by the fact that "*solely vorticity cannot distinguish between swirling motions and shearing motions*", Kida and Miura [1] (this fact is emphasized also by Jeong and Hussain [2] and Cucitore, Quadrio and Baron [3]). Let us add an analogous statement for strain rate: "*solely strain rate cannot distinguish between straining motions and shearing motions*". Vorticity and strain rate reflect the conventional Cauchy-Stokes decomposition of the local relative motion near a point into two elementary homogeneous motions: a pure irrotational straining motion along principal axes of the rate of strain tensor (generally including a uniform dilatation) and a rigid-body rotation. In Kolář [4] an arbitrary instantaneous state of the local relative motion near a point is decomposed into three elementary motions–each described by an additive part of $\nabla \mathbf{u}$ with a distinct tensor character–explicitly including an *effective* pure shearing motion. The first two elementary parts of the proposed triple decomposition remind, at least in tensor character, the two elementary parts of the conventional double decomposition, $\nabla \mathbf{u} = \mathbf{S} + \boldsymbol{\Omega}$, and represent their *residual* portions after extracting an *effective* pure shearing motion.

Triple Decomposition of Motion

The algorithm of the triple decomposition of motion is briefly summarized below. For details, justification and discussion see Kolář [4] where three elementary motions are introduced in terms of a *virtually structured continuum*. The triple decomposition of the local relative motion near a point aims, basically, at the extraction of an *effective* pure shearing motion. It reads

$$\nabla \mathbf{u} = (\nabla \mathbf{u})_{\mathrm{EL}} + (\nabla \mathbf{u})_{\mathrm{RR}} + (\nabla \mathbf{u})_{\mathrm{SH}} \tag{1}$$

where the three elementary homogeneous motions are described by additive parts of $\nabla \mathbf{u}$: a pure irrotational straining motion given by symmetric tensor $(\nabla \mathbf{u})_{\mathrm{EL}}$ (subscript "EL" reminds the term "elongation"), a rigid-body rotation given by antisymmetric tensor $(\nabla \mathbf{u})_{\mathrm{RR}}$, and an *effective* pure shearing motion $(\nabla \mathbf{u})_{\mathrm{SH}}$ described below.

A general pure shearing motion (that is, at this stage, without specification of the term "*effective*") is defined by a "purely asymmetric tensor form" of $\nabla \mathbf{u}$ fulfilling *in a suitable reference frame* (null tensors are excluded, the subscript $/j$ denotes differentiation)

$$u_{i/j} = 0 \quad OR \quad u_{j/i} = 0 \quad \text{(for all } i, j\text{)}. \tag{2}$$

The desired reference frame, showing an *effective* pure shearing motion "in a clearly visible manner" described by the form (2), is called a *basic reference frame* (BRF). In this frame, the local relative motion is decomposed in terms of additive parts of $\nabla \mathbf{u}$ at a given point. In any other frames rotated with respect to the *basic reference frame* under an orthogonal transformation $\mathbf{Q}$, an arbitrary elementary part of $\nabla \mathbf{u}$, say $\mathbf{A}$, is described simply by $\mathbf{QAQ}^{\mathrm{T}}$. Note that the sum of additive parts $\sum_i \mathbf{A}_i$ transforms as

$$\mathbf{Q}\left(\sum_i \mathbf{A}_i\right)\mathbf{Q}^{\mathrm{T}} = \sum_i \mathbf{QA}_i\mathbf{Q}^{\mathrm{T}} . \tag{3}$$

The triple decomposition then reads in the BRF (specified afterwards)

$$\begin{aligned}
\nabla \mathbf{u} &\equiv \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix}^{\mathrm{BRF}} \\
&= \begin{pmatrix} u_x & (\operatorname{sgn} u_y)\mathrm{MIN}(|u_y|,|v_x|) & \bullet \\ (\operatorname{sgn} v_x)\mathrm{MIN}(|u_y|,|v_x|) & v_y & \bullet \\ \bullet & \bullet & w_z \end{pmatrix}^{\mathrm{BRF}} \\
&\quad + \begin{pmatrix} \textit{purely} \\ \textit{asymmetric} \\ \textit{matrix} \end{pmatrix}^{\mathrm{BRF}} \\
&= [\text{tensor core of } (\nabla \mathbf{u})]^{\mathrm{BRF}} + [\text{tensor overhang of } (\nabla \mathbf{u})]^{\mathrm{BRF}} \\
&= [(\nabla \mathbf{u})_{\mathrm{EL}} + (\nabla \mathbf{u})_{\mathrm{RR}}] + (\nabla \mathbf{u})_{\mathrm{SH}}
\end{aligned} \tag{4}$$

where (using simplified notation) subscripts x, y, z stand for spatial partial derivatives, and the remaining two non-specified pairs of off-diagonal terms in the first matrix of the decomposition are constructed strictly analogously as the specified one. The introduced frame-dependent *tensor-core matrix* is characterized by the symmetry in absolute values of components. This matrix determined above in the BRF is further decomposed into two parts as it represents a sum $[(\nabla \mathbf{u})_{\mathrm{EL}} + (\nabla \mathbf{u})_{\mathrm{RR}}]$. Note that each pair of off-diagonal terms of the *tensor-core matrix* is either symmetric or antisymmetric. The introduced frame-dependent *tensor-overhang matrix* has a purely asymmetric tensor form defined by (2) and represents in the BRF an *effective* pure shearing motion.

The BRF is determined from the condition

$$\left\| [\text{tensor core of } (\nabla \mathbf{u})]^{\mathrm{BRF}} \right\| = \text{MIN over all frames} \tag{5}$$

where the symbol $\| \ldots \|$ denotes a standard tensor norm (frame-independent absolute tensor value). Alternatively, in terms of the Cauchy-Stokes double decomposition of $\nabla \mathbf{u}$ (e.g. Truesdell and Toupin [5], Batchelor [6])

$$\nabla \mathbf{u} = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^{\mathrm{T}}\right) + \frac{1}{2}\left(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathrm{T}}\right) \equiv \mathbf{S} + \boldsymbol{\Omega} , \tag{6}$$

the BRF is determined from the condition

$$\left[\left|S_{12}\Omega_{12}\right| + \left|S_{23}\Omega_{23}\right| + \left|S_{31}\Omega_{31}\right|\right]^{\mathrm{BRF}} = \text{MAX over all frames} \; . \quad (7)$$

It is clearly seen from (7) that the BRF, unlike the system of principal axes of $\mathbf{S}$, is determined *simultaneously* on the basis of a non-zero $\mathbf{S}$ and a non-zero $\mathbf{\Omega}$.

Further, from the viewpoint of the double decomposition the *effective* pure shearing motion $(\nabla\mathbf{u})_{\mathrm{SH}}$ itself (i.e. separately) represents a certain coupling of a pure irrotational straining motion with a rigid-body rotation. Consequently, the *triple* decomposition of $\nabla\mathbf{u}$ may be substituted by the *quadruple* decomposition of $\nabla\mathbf{u}$ with two different (symmetric) strain-rate terms and two different (antisymmetric) vorticity terms

$$\begin{aligned}\nabla\mathbf{u} &= (\nabla\mathbf{u})_{\mathrm{EL}} + (\nabla\mathbf{u})_{\mathrm{RR}} + (\nabla\mathbf{u})_{\mathrm{SH}} \\ &= (\nabla\mathbf{u})_{\mathrm{EL}} + (\nabla\mathbf{u})_{\mathrm{RR}} \\ &\quad + \frac{1}{2}\left[(\nabla\mathbf{u})_{\mathrm{SH}} + \left((\nabla\mathbf{u})_{\mathrm{SH}}\right)^{\mathrm{T}}\right] + \frac{1}{2}\left[(\nabla\mathbf{u})_{\mathrm{SH}} - \left((\nabla\mathbf{u})_{\mathrm{SH}}\right)^{\mathrm{T}}\right] \\ &= \mathbf{S}_{\mathrm{RES}} + \mathbf{\Omega}_{\mathrm{RES}} + \mathbf{S}_{\mathrm{SH}} + \mathbf{\Omega}_{\mathrm{SH}} \\ &= \left[\mathbf{S}_{\mathrm{RES}} + \mathbf{S}_{\mathrm{SH}}\right] + \left[\mathbf{\Omega}_{\mathrm{RES}} + \mathbf{\Omega}_{\mathrm{SH}}\right] = \mathbf{S} + \mathbf{\Omega} \; . \end{aligned} \quad (8)$$

These four terms (in the last two square brackets) are labelled "*residual* strain rate" and "*shear* strain rate", "*residual* vorticity" and "*shear* vorticity". Note that in the triple decomposition, $\mathbf{S}$ and $\mathbf{\Omega}$ are cut down in magnitudes to "share" their portions through the third term $(\nabla\mathbf{u})_{\mathrm{SH}}$ associated with a pure shearing motion as

$$\nabla\mathbf{u} = \mathbf{S}_{\mathrm{RES}} + \mathbf{\Omega}_{\mathrm{RES}} + (\nabla\mathbf{u})_{\mathrm{SH}} \; . \quad (9)$$

Triple Decomposition of Motion in 2D Fluid Flow

The nature of the proposed triple decomposition (and its physical justification) is straightforward in 2D fluid motion. To make the decomposition of motion clearly visible, the conventional double decomposition frequently uses as a reference frame the *shear-free frame* of principal axes of the strain-rate tensor. A uniform dilatation can be removed prior to further analysis of $\nabla\mathbf{u}$ without loss of generality and an arbitrary 2D flow can be described (using simplified notation) in the system of principal axes by the form

$$\nabla\mathbf{u} = \begin{pmatrix} s & -\omega & 0 \\ \omega & -s & 0 \\ 0 & 0 & 0 \end{pmatrix}^{\text{PRINCIPAL AXES}} . \quad (10)$$

By a suitable rotation of coordinate axes we approach the desired BRF. For 2D rotational motion, the BRF corresponds to the *shear frame* in which the deviatoric strain-rate tensor has zeros on the leading diagonal and $\nabla\mathbf{u}$ is given by

$$\nabla\mathbf{u} = \begin{pmatrix} 0 & s-\omega & 0 \\ s+\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^{\mathrm{BRF}} . \quad (11)$$

In this frame, in the plane of 2D motion, there are only two characteristic "shearing interactions" of $u_y\mathrm{d}y$ with $v_x\mathrm{d}x$, namely with the same or opposite rotational orientations, see Fig. 1. For both orientations, the magnitude of the resulting "superimposed" pure shearing motion is given simply by the difference of absolute values of u_y and v_x.

The present new approach is easy to demonstrate *geometrically* in 2D fluid motion in terms of the decomposition of vorticity ω and strain rate s, see (10), (11), as shown in Fig. 1. In this figure the vorticity and strain-rate components are proportional to the infinitesimal changes of related characteristic angles during the infinitesimal change of time. In Fig. 1(a) the characteristic angles correspond to the *residual* vorticity ω_{RES} (associated with the rigid-body rotation) and *shear* vorticity ω_{SH} (associated with the pure shearing motion) while their sum is proportional to the *total* vorticity ω. In Fig. 1(b) the characteristic angles correspond to the *residual* strain rate s_{RES} and *shear* strain rate s_{SH} while their sum is proportional to the *total* strain rate s. This "superimposing" geometrical construction is of virtual nature and applicable to infinitesimal motional changes only. However, it provides a clear qualitative insight and interpretation of the vorticity and strain-rate components.

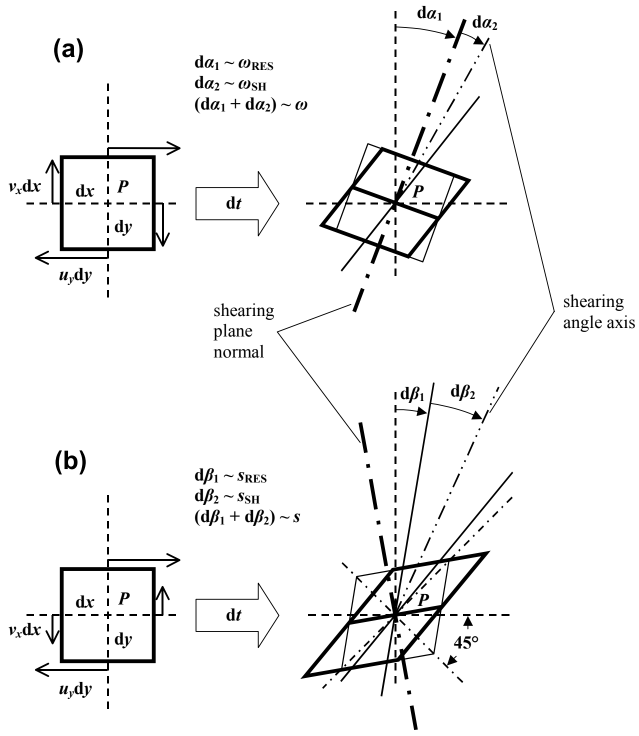

Figure 1 Geometrical interpretation of the triple decomposition in 2D fluid motion: (a) vorticity components, (b) strain-rate components.

Fig. 1 indicates two significant inherent features of the proposed decomposition. Firstly, the concept of the triple decomposition as defined by (4) implies (generally in 3D) that the corresponding components of the *residual* vorticity and *shear* vorticity have the *same signs in the BRF*. The same holds for the *residual* strain rate and *shear* strain rate. Secondly, Fig. 1 indicates a certain "principle of exclusivity" which is valid in 2D. The principle of exclusivity may be expressed for arbitrary 2D velocity fields representing an isochoric part of motion as

$$(\nabla\mathbf{u})_{\mathrm{EL}}^{\mathrm{2D}} = \mathbf{0} \quad OR \quad (\nabla\mathbf{u})_{\mathrm{RR}}^{\mathrm{2D}} = \mathbf{0} \quad (12a)$$

or, alternatively, in terms of ω and s as

$$s_{\mathrm{RES}} = 0 \quad OR \quad \omega_{\mathrm{RES}} = 0 \; . \quad (12b)$$

The expressions (12*a*, *b*) say that a non-zero *residual* vorticity apparently existing only for the same rotational orientations of $u_y\mathrm{d}y$ and $v_x\mathrm{d}x$, see Fig. 1, excludes the existence of a non-zero *residual* strain rate existing only for the opposite rotational orientations of $u_y\mathrm{d}y$ and $v_x\mathrm{d}x$. Consequently, Fig. 1(a) represents an inevitable reduction of the three distinct terms of the triple decomposition to the form $(\nabla\mathbf{u})_{\mathrm{RR}} + (\nabla\mathbf{u})_{\mathrm{SH}}$ while Fig. 1(b) represents the reduction to $(\nabla\mathbf{u})_{\mathrm{EL}} + (\nabla\mathbf{u})_{\mathrm{SH}}$.

Components of the triple decomposition of the local relative motion near a point and corresponding flow patterns for various flow situations in 2D isochoric fluid motion are shown in Fig. 2. All possible flow configurations near a point for fixed u_y and variable v_x are depicted in the corresponding BRFs. The points

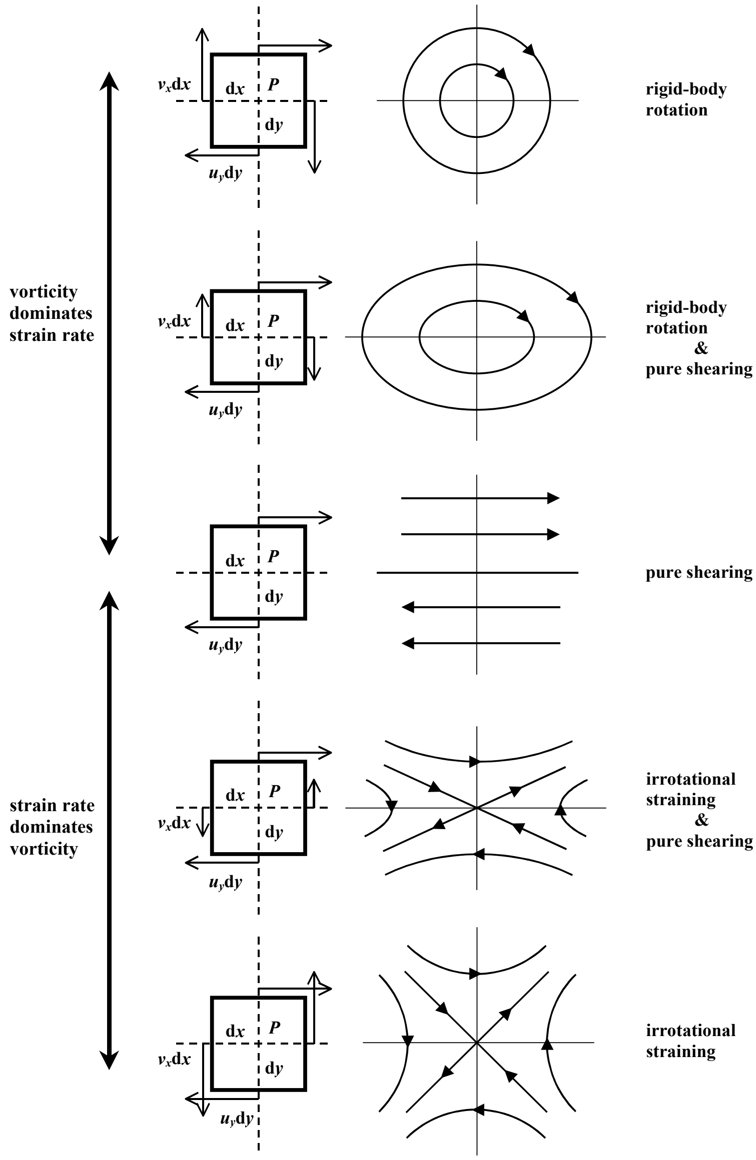

Figure 2 Components of the triple decomposition of motion and flow patterns for various flow situations in 2D fluid motion.

itself can be described as critical points and the local flow patterns correspond to the leading terms of a Taylor series expansion for the velocity field in terms of space coordinates (Perry and Chong [7], Chong, Perry and Cantwell [8]).

Application of the Triple Decomposition of Motion to 2D Velocity-Field Analysis

Let us remind that a uniform dilatation can be removed prior to further analysis of $\nabla \mathbf{u}$ without loss of generality and applicability to compressible flows. In terms of s and ω introduced in (10) and (11) by a suitable rotation of coordinate axes

$$\begin{pmatrix} u_x & u_y & 0 \\ v_x & -u_x & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} s & -\omega & 0 \\ \omega & -s & 0 \\ 0 & 0 & 0 \end{pmatrix}^{\text{PR. AXES}} \to \begin{pmatrix} 0 & s-\omega & 0 \\ s+\omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^{\text{BRF}}$$

where

$$|s| = \left(\sqrt{4u_x^2 + \left(u_y + v_x\right)^2}\right)/2 \,, \tag{13}$$

$$\omega = \left(v_x - u_y\right)/2 \,, \tag{14}$$

and their *residual* and *shear* components (note that these are of the *same signs in the BRF*) we can draw from (4) and (8) the following set of relations

$$s = s_{\text{RES}} + s_{\text{SH}} \,, \tag{15}$$

$$|s| = |s_{\text{RES}}| + |s_{\text{SH}}| \,, \tag{16}$$

$$s_{\text{SH}} = (\operatorname{sgn} s)|\omega| \quad \textit{for} \quad |s| > |\omega| \,, \tag{17}$$

$$s_{\text{SH}} = s \quad \textit{for} \quad |s| < |\omega| \,, \tag{18}$$

$$s_{\text{RES}} = s - s_{\text{SH}} \quad \textit{for} \quad |s| > |\omega| \,, \tag{19}$$

$$s_{\text{RES}} = s - s_{\text{SH}} = 0 \quad \textit{for} \quad |s| < |\omega| \,, \tag{20}$$

$$\omega = \omega_{\text{RES}} + \omega_{\text{SH}} \,, \tag{21}$$

$$|\omega| = |\omega_{\text{RES}}| + |\omega_{\text{SH}}| \,, \tag{22}$$

$$\omega_{\text{SH}} = \omega \quad \textit{for} \quad |s| > |\omega| \,, \tag{23}$$

$$\omega_{\text{SH}} = (\operatorname{sgn} \omega)|s| \quad \textit{for} \quad |s| < |\omega| \,, \tag{24}$$

$$\omega_{\text{RES}} = \omega - \omega_{\text{SH}} = 0 \quad \textit{for} \quad |s| > |\omega| \,, \tag{25}$$

$$\omega_{\text{RES}} = \omega - \omega_{\text{SH}} \quad \textit{for} \quad |s| < |\omega| \,. \tag{26}$$

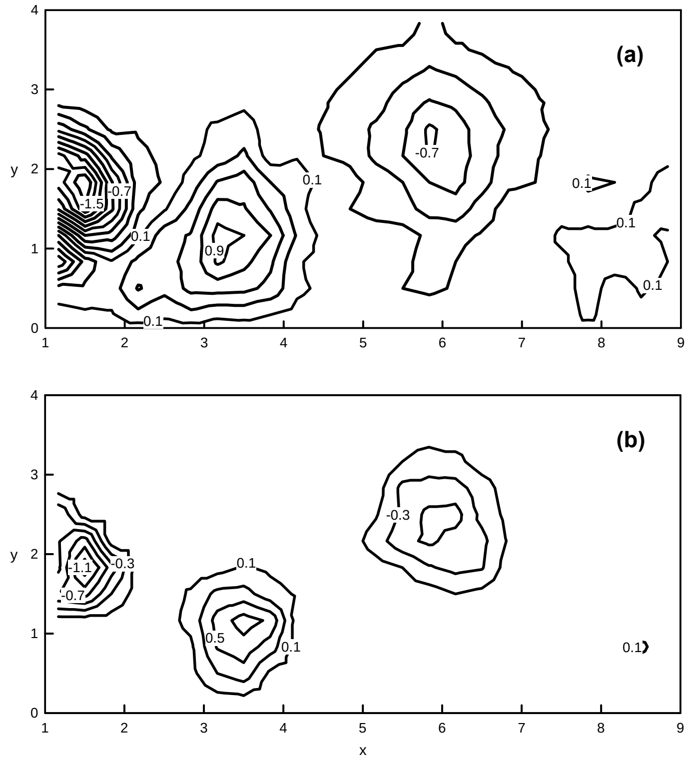

Figure 3 Comparison of (a) total vorticity and (b) *residual* vorticity for plane turbulent wake (velocity data taken from [9], phase 3, contour int. 0.2).

The case $|s| = |\omega|$ represents a simple shear, hence an *effective* pure shearing motion is the only non-zero component of the triple decomposition of motion in this case.

A practical application of the present approach to the velocity data of the *nominally* plane turbulent wake of two side-by-side square cylinders by Kolář, Lyn and Rodi [9] is shown in Fig. 3.

It is clearly seen from Fig. 3 that by removing the shearing component of motion, that is after the extraction of an *effective* pure shearing motion by means of the triple decomposition of motion, we reach a more adequate picture of vortical structures. These structures are characterized exclusively by swirling motion in terms of the *residual* vorticity which is non-zero for $|s| < |\omega|$ and determined from (24) and (26) using (13) and (14). The *residual* vorticity appears as the proper kinematic quantity to identify true vortex cores determining both their boundary and inner structure.

Similarly we can distinguish the *residual* strain rate from the total strain rate, for example, while describing the process of turbulence production in the saddle regions of the primary large-scale vortical structures in turbulent wakes.

Conclusions

2D flow fields have been described in terms of the new triple decomposition of the local relative motion near a point. The triple decomposition of motion aims at the extraction of an *effective* pure shearing motion which is responsible for a specific portion of vorticity labelled "*shear* vorticity" and for a specific portion of strain rate labelled "*shear* strain rate". The triple decomposition of motion is closely associated with the so-called *basic reference frame* (BRF) where it is performed. Some fundamental issues have been pointed out: (i) *principle of exclusivity* holds between 2D *residual* vorticity and 2D *residual* strain rate for an isochoric part of motion, (ii) *residual* and *shear* components of both vorticity and strain rate have the *same signs in the BRF*. Considering arbitrary 2D velocity fields, simple relations have been derived for vorticity and strain-rate components.

The kinematic structure inferred from the triple decomposition of motion should help in the description and analysis of a wide variety of fluid-dynamical processes and fluid-flow phenomena. It may prove its usefulness for flow classification schemes as well as for complex rheological problems.

Acknowledgments

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References

[1] Kida, S. & Miura, H., Identification and analysis of vortical structures, *Eur. J. Mech. B/Fluids*, **17**, 1998, 471-488.

[2] Jeong, J. & Hussain, F., On the identification of a vortex, *J. Fluid Mech.*, **285**, 1995, 69-94.

[3] Cucitore, R., Quadrio, M. & Baron, A., On the effectiveness and limitations of local criteria for the identification of a vortex, *Eur. J. Mech. B/Fluids*, **18**, 1999, 261-282.

[4] Kolář, V., Triple decomposition of the local relative motion near a point, submitted to *J. Fluid Mech.*

[5] Truesdell, C. & Toupin, R.A., The Classical Field Theories, in *Encyclopedia of Physics, Vol. III/1, Principles of Classical Mechanics and Field Theory*, Flügge, S. (editor), Springer-Verlag, 1960.

[6] Batchelor, G.K., *An Introduction to Fluid Dynamics*. Cambridge University Press, 1967.

[7] Perry A.E. & Chong M.S., A description of eddying motions and flow patterns using critical-point concepts, *Annu. Rev. Fluid Mech.*, **19**, 1987, 125-155.

[8] Chong, M.S., Perry, A.E. & Cantwell, B.J., A general classification of three-dimensional flow fields, *Phys. Fluids A* **2**, 1990, 765-777.

[9] Kolář, V., Lyn, D.A. & Rodi, W., Ensemble-averaged measurements in the turbulent near wake of two side-by-side square cylinders, *J. Fluid Mech.* **346**, 1997, 201-237.