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A new solution for ocean waves propagating over a sloping beach

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Abstract

In this paper, the phenomenon of ocean waves propagating over a sloping beach is re-examined. Unlike previous analytical approximations, we propose an exact solution for a general case with arbitrary beach shape. Two different beach shapes are used as numerical examples. Numerical results demonstrate the significant influence of beach shapes on the water surface elevation and velocity.

Introduction

The phenomenon of ocean waves transformation from offshore (deep water) to nearshore (shallow water) is particularly important for the design and protection of the coastline. This includes the topics of wave breaking, the stability of the coastline, and beach nourishment. Also, the transformation of wave energy in the nearshore region is a dominant factor in the design of coastal structures.

Since the perturbation technique was first applied to the development of ocean waves [12], the symmetric ocean waves in a uniform water depth has been widely studied since the 1980s. With the development of computational technique, the wave phenomenon in a uniform depth is no longer a mystery. However, in realistic environments, the seabed bottom is varied, the variation of the seabed bottom will affect the free surface fluctuation from deep to shallow water.

Numerous investigations for the ocean waves propagating over a sloping seabed have been carried out. Carrier and his co-author [1, 2] developed a series of analytical solutions for gravity waves propagating on water of variable depth. Their solutions have been limited to a beach with constant slope, although the solution for a beach with arbitrary bottom was suggested.

To date, the commonly used model for waves propagating over a sloping seabed is based on the wave model of a uniform depth, and then apply the conservation of energy flux to solve the wave fluctuation with varying depth step by step [3, 8]. This type of approaches cannot represent the effects of seabed bottom slope in the solution. A few researchers have attempted to directly take into account of slope in the whole problem [7]. However, their approaches only limited to the cases with a small slope and a small relative water depth. Recently, with advance of numerical schemes, the wave propagating over a sloping seabed even to wave breaking state can be solved numerically. For example, the parabolic wave model proposed by Li [9] has been widely used and extended to various situations [4, 10, 13]

Besides analytical approximations, significant advances have been made in developing mathematical models to describe fully non-linear and weakly dispersive waves propagating over an impermeable bottom [5, 6, 11]. Based on the inviscid fluid assumption, these models reduce the three-dimensional Euler equations to a set of two-dimensional governing equations, These equations are usually expressed in terms of the free surface displacements and representative horizontal velocity components, which are either evaluated at a certain elevation, or depth averaged.

In this study, a new analytical solution is developed for the phenomenon of ocean wave propagating over a sloping beach. Unlike previous analytical approximations, we consider the beach with an arbitrary shape, rather than linear beach. Two types of beach shapes are used as examples, and their effects on the water surface elevation will be investigated.

Theoretical Formulations

Boundary Value Problem

In this study, we consider the the ocean gravity waves propagating over a sloping beach, as depicted in figure 1. In the figure, h_a is the reference water depth far from the beach, $h(x,t)$ is the water surface elevation, which is defined by

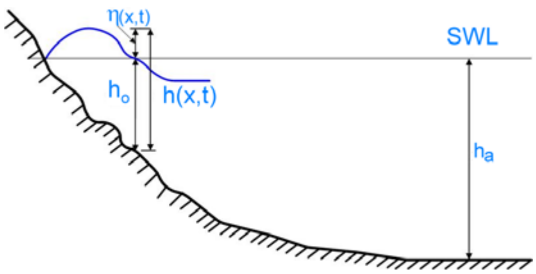


Figure 1: Geometry of the general propagation problem.

$$h(x,t) = h_o(x) + \eta(x,t), \tag{1}$$

where $\eta(x,t)$ represents the fluctuation of water wave, and $h_o(x)$ is the water depth at the location (x).

Based on shallow water theory [1], the governing equations for the ocean waves propagating over an incompressible and invisid fluid can be expressed in Eulerian system as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = 0, \tag{2a}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} = 0, \tag{2b}$$

where u is the velocity in the horizontal direction, t is the time, g is the gravitational acceleration, and the subscripts ”x” and ”t” denote the partial differentiation respective to x and t, respectively.

Now, we consider the problem in a Lagrangian system, and choose h_a a the reference height, the relationship between two co-ordinates is

$$\frac{\partial x}{\partial X}=\frac{h_a}{h(x,t)}=\frac{h_a}{h_o(x)+\eta(x,t)}. \tag{3}$$

To simplify the problem, we linearise (3) as,

$$\frac{\partial x}{\partial X}=\frac{h_a}{h(x,t)}\approx\frac{h_a}{h_o(x)}. \tag{4}$$

Then, the linearised governing equations in a Lagrangian system can be expressed as

$$\frac{\partial u}{\partial t} + g\frac{h_o}{h_a}\frac{\partial \eta}{\partial X}=0, \tag{5a}$$
$$\frac{\partial \eta}{\partial t} + \frac{h_o^2}{h_a}\frac{\partial u}{\partial X}=0. \tag{5b}$$

To simplify the mathematical expressions, we non-dimensionalise the whole problem with the following variables,

$$(X^*,\eta^*,h)=\frac{(X,\eta,h)}{h_a},\quad t^*=\frac{t}{\left(h_a/\sqrt{gh_a}\right)}, \tag{6}$$

where the superscript "$*$" denote the non-dimensional parameters. To avoid the complicated mathematical expressions, the "*" will be ignored, and all physical variables are non-dimensional parameters in the following section, unless specified.

Introducing (6) into (5a) and (5b), the governing equations can be re-written as

$$\frac{\partial u}{\partial t} + h_o\frac{\partial \eta}{\partial X}=0 \tag{7a}$$
$$\frac{\partial \eta}{\partial t} + h_o^2\frac{\partial u}{\partial X}=0 \tag{7b}$$

Analytical Solution

In this paper, we attempt to solve the above governing equations (7a) and (7b) analytically. Herein, we define a new variable,

$$\frac{dR}{dX}=\frac{1}{h_o^{3/2}} \tag{8}$$

which leads to

$$\frac{\partial u}{\partial t} + \frac{1}{\sqrt{h_o}}\frac{\partial \eta}{\partial R} \tag{9a}$$
$$\frac{\partial \eta}{\partial t} + \sqrt{h_o}\frac{\partial u}{\partial R}=0 \tag{9b}$$
$$\frac{\partial^2 \eta}{\partial t^2} - \sqrt{h_o}\frac{\partial}{\partial R}\left(\frac{1}{\sqrt{h_o}}\frac{\partial \eta}{\partial R}\right)=0 \tag{9c}$$

Let $C(\varepsilon)=\sqrt{h_o(X)}$, we have

$$\frac{\partial^2 \eta}{\partial t^2} - C(R)\frac{\partial}{\partial R}\left(\frac{1}{C(R)}\frac{\partial \eta}{\partial R}\right)=0 \tag{10a}$$
$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{C(R)}\frac{\partial}{\partial R}\left(C(R)\frac{\partial u}{\partial R}\right)=0 \tag{10b}$$

in which

$$R=\int_0^X\frac{ds}{C^3(s)}\quad\text{and}\quad X=\int_0^R C^3(s)ds \tag{11}$$

Note that (10a) contains the *beach shape function*, $C(R)=\sqrt{h_0(X)}$, which describes the variation in the beach profile with depth. In general, it is difficult to obtain analytical solutions for equations of the form (10a), However, an exact solution is possible using the approach of Varley and Seymour [14].

The general solution for (10a) can be expressed as

$$\eta = \sum_{n=0}^{N} f_n(R)\frac{\partial^{N-n}F}{\partial R^{N-n}} \tag{12a}$$
$$u = \sum_{n=0}^{N} e_n(R)\frac{\partial^{N-n}E}{\partial R^{N-n}} \tag{12b}$$

where $f_o=1/e_o=\sqrt{C(R)}=h_o^{1/4}(X)$, and E and F satisfy

$$\frac{\partial E}{\partial t}+\frac{\partial F}{\partial R}=0\quad\text{and}\quad\frac{\partial F}{\partial t}+\frac{\partial E}{\partial R}=0 \tag{13}$$

The function E and F can be expressed as

$$E = A(t+R)+B(t-R) \tag{14a}$$
$$F = -A(t+R)+B(t-R) \tag{14b}$$

where $A(t+R)$ is given as the incident wave components, and $B(t-R)$ is an unknown function, which needs to be determined later.

To find B, the following boundary conditions are required:

$$\eta \to 0 \quad\text{as}\quad C(R)\to 0 \tag{15a}$$
$$u \quad\text{bounded}\quad\text{as}\quad C(R)\to 0 \tag{15b}$$

Using $N=1$ as the first approximation, we have

$$\eta = \sqrt{C(R)}\frac{\partial F}{\partial R}+\ell_1 F \tag{16a}$$
$$u = \frac{1}{\sqrt{C(R)}}\frac{\partial E}{\partial R}+k_1 E \tag{16b}$$

With the boundary conditions, we have $A(t)=B(t)$, which gives us

$$E = A(t+R)+A(t-R) \quad (17a)$$
$$F = -A(t+R)+A(t-R) \quad (17b)$$

If we consider the incident wave A(x,t) as

$$A(x,t) = A_0 \cos(\frac{2\pi}{L}(x-ct)) = A_0 \cos(\frac{2\pi h_a}{L}(X^* - t^*)) \quad (18)$$

where L is the wavelength of incident wave in deep water, A_0 is the amplitude of waves.

Numerical Examples

In this paper, two types of beach profiles are used:

$$\text{case I:} \quad h_o(R) = [0.5\tanh(0.1R)]^4 \quad (19a)$$
$$\text{case II:} \quad h_o(R) = R^4 \quad (19b)$$

Two different beach shapes are plotted in figure 2. As shown in the figure, Case I represents a case with gentle slope, which is a function of tanh, while Case II represents a case of rapidly slope, which is a function of $R(x)^4$. In this study, we use h_a=0.05L as the reference water depth, because we are only concern with the case of shallow water. In the following discussion, the non-dimensional variables will be represented with "*" superscripts.

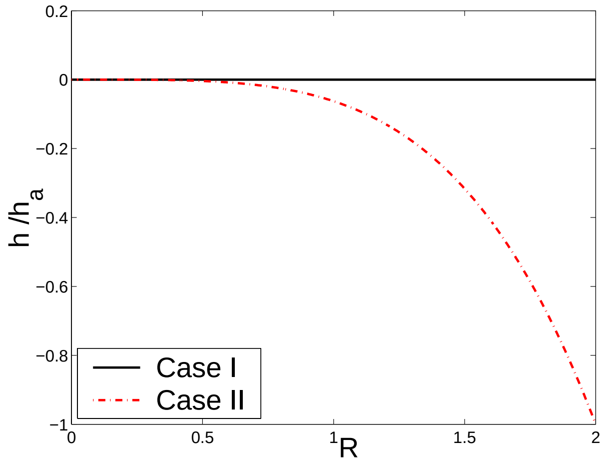
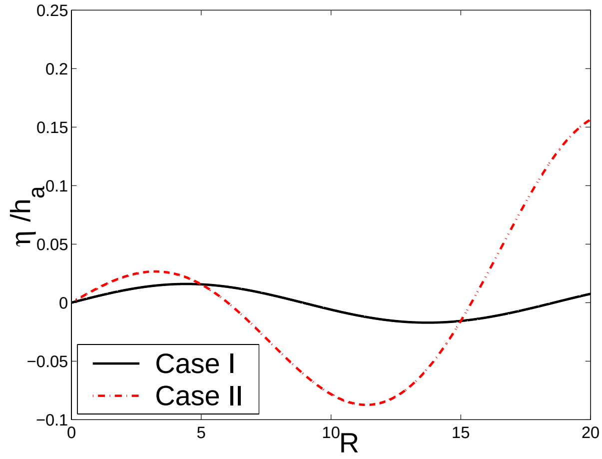


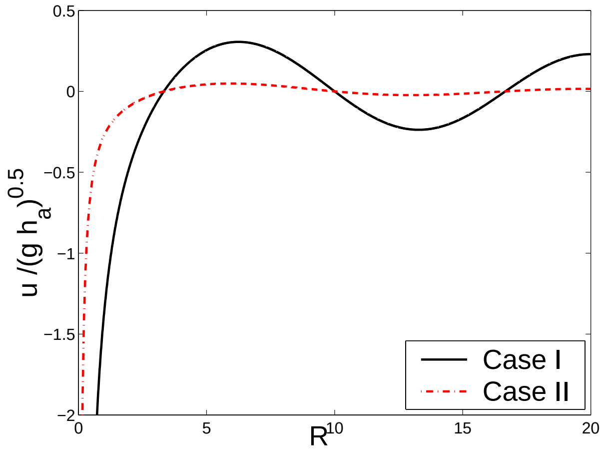
Figure 2: Two types of beach profiles.

With the beach profile shown in figure 2, the water surface elevations (η/h_a) with different beach profile are illustrated in figure 3(a). As shown in the figure, functions of beach profiles significantly affect the water surface elevation (η). The distribution of velocity versus horizontal distance (x/h_a) is illustrated in figure 3(b). Again, significant influence of beach profile is found.

it is noted that the amplitude of the water surface elevation ($|\eta|$) slowly increases as horizontal distance (x) increases. It is because the present solution is only valid for the shallow water, i.e., $h_a/L < 0.05$, which is based on the assumption of shallow water expansion. For the cases with intermediate water and deep water, the conventional Stokes wave theory should be used.



(a) water surface elevation (η)



(b) velocity (u)

Figure 3: Comparison of (a) water surface elevation and (b) velocity with different beach profiles. (T=0.5)

Conclusions

In this paper, an exact solution for ocean wave propagating over a sloping beach is derived. In the new analytical solution, an arbitrary beach shape is considered, which has been assumed to be a linear function in the past. Two different beach profiles are used as numerical examples. Numerical results demonstrate the significant effects of beach profile on the water surface elevation and velocity profiles.

Acknowledgements

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